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The work presents the second part of the second edition of its previous one published in 2000 under the same title, containing the proof (in ZF) of the inaccessible cardinals nonexistence, which is enriched and improved now.

This part contains applications of the subinaccessible cardinals apparatus and its basic tools – theories of reduced formula spectra and matrices, disseminators and others, which are used here in this proof and are set forth now in their more transparent and simplified form. Much attention is devoted to the explicit and substantial development and cultivation of basic ideas, serving as grounds for all main constructions and reasonings. The proof of the theorem about inaccessible cardinals nonexistence is presented in its detailed exposition. Several easy consequences of this theorem and some well-known results are presented.

Ref. 38.

R e f e r e e s:

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To my mother Ann

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Introduction

This work represents the direct continuation of its previous Part I [27] and constitutes the uniform text with it. The author has considered it natural to organize this work in such a manner, that it continues all enumerations of [27] and among them enumerations of chapters, paragraphs, definitions, statements and formulas and even enumerations of comments and references as well.

Also formulas, notions or symbols used in this work without explanations have been already introduced in Part I [27] or generally accepted or used in remarkable text of Jech [18] “Lectures in Set Theory with Particular Emphasis on the Method of Forcing”, providing many basic notions and information and much more; therefore here they are assumed to be known and will be used frequently without comments.

So, it would be more convenient for the reader to familiarize preliminarily with [27] and with the main notions and denotations of this work at least in outline.

Anyway, it would be useful to get acquainted with the plan of all the work beforehand and with the outline of developing basic ideas as they are presented in [27] on pp. 9–18.

With this agreement in hand, the reader should remember that all backward references to paragraphs with numbers less than 7 concern Part I [27], analogously for enumerated statements and so on.

As to the content of this work and to the technical side of the matter it should be noticed that it follows its previous edition [17]

of 2000, but in the more systematic way.

Also it should be pointed out, that in this edition [17] and foregoing works the author tendered to avoid the usage of new terminology, especially private notions and symbols of his own, over certainly necessary, because he was uneasy about difficulties and unacceptance it may cause for the reader.

Nevertheless, the perception of the previous edition [17] by the readers showed, that apprehensions of this kind are inappropriate and, so, this usage is unavoidable all the same.

Therefore in the present work the author has taken another stand and has considered more natural to involve all the system of his own private concepts and definitions in the whole, which he has developed since 1976, because it possesses the technical and conceptual expressiveness and leads to the very essence of the matter and, so, it would be too artificial to avoid its usage (see comment 3 [27] as an example). Some statements have received their strengthening; some details considered obvious in the author's previous works, here have received their refinement; some arguments have changed their places for the more suitable; also some suitable redesignations are used.

But as for main constructions, one should note, that the present work follows the edition [17] of 2000 but in the more clarified way; the main result of this work: the system

$$ZF + \exists k \text{ (} k \text{ is weakly inaccessible cardinal)}$$

is inconsistent; all the reasonings are carried out in this theory. All weakly inaccessible cardinals become strongly inaccessible in the constructive class L and therefore the reasonings are carried over to the standard countable basic model

$$\mathfrak{M} = (L_{\chi^0}, \in, =)$$

of the theory

$$ZF + V = L + \exists k \text{ (} k \text{ is weakly inaccessible cardinal),}$$

and further k is the smallest inaccessible cardinal in \mathfrak{M} . Actually only the formulas of the limited length from this theory are used; moreover, the countability of this model is required only for some technical convenience and it is possible to get along without it (see “Preliminaries” [27]). In this model \mathfrak{M} the so called matrix functions are constructed, possessing simultaneously the two properties of monotonicity and nonmonotonicity, that provides

Main theorem (ZF)

There are no weakly inaccessible cardinals.

It implies the nonexistence of strongly inaccessible cardinals and therefore nonexistence of all other large cardinals. These matrix functions are constructed and treated by means of the elementary language from the formula classes (see definition 2.1 [27]) of some fixed level > 3 over the standard model

$$(L_k, \in, =)$$

and further all constructions are carried out by means of this language (if the opposite is not pointed out by the context).

In addition in §12 some easy consequences of the Main theorem and some well-known results are presented.

Chapter II

Special Theory: Matrix Functions

7 Matrix δ -Functions

Here we are going to start the further development of the idea of the Main theorem proof and to modify the simplest matrix functions $S_{\chi f}^{<\alpha_1}$ (see definition 5.14 [27]) in such a way that their new specialized variants – the so called α -functions – will provide the required contradiction: they will possess the property of \leq -monotonicity and at the same time will be deprived of it.

Let us remind that the simplest matrix functions, which were considered in § 5 [27], possess the property of monotonicity, but it came out that the direct proof of the required contradiction – the proof of their nonmonotonicity – is hampered by the following obstacle: some essential properties of lower levels of universe do not extend up to jump cardinals of matrices on their carriers, which are values of the matrix functions.

In order to destroy this obstacle we shall equip such matrices with their corresponding disseminators and as the result the simplest matrix functions will be transformed to their more complicated forms, α -function.

However, the direct formation of these functions seems to be considerably complicated and some their important singularities unmotivated.

Therefore in order to represent their introduction in the more transparent way we shall beforehand undertake the second ap-

proach to the idea of the Main theorem proof and turn attention to their more simple forms, that is to the δ -functions.

To this end we shall apply results of §6 [27] for $m = n + 1$ and the fixed level $n > 3$, but the notion of disseminator matrix should be sharpened; all disseminators in what follows will be of the level $n + 1$ (see definition 6.9 [27]).

Definition 7.1

Let

$$\gamma < \alpha < \alpha_1 \leq k.$$

1) We denote through $\mathbf{K}_n^{\forall < \alpha_1}(\gamma, \alpha)$ the formula:

$$SIN_{n-1}^{< \alpha_1}(\gamma) \wedge \forall \gamma' \leq \gamma (SIN_n^{< \alpha_1}(\gamma') \longrightarrow SIN_n^{< \alpha}(\gamma')) .$$

If this formula is fulfilled by the constants γ, α, α_1 , then we say that α conserves $SIN_n^{< \alpha_1}$ -cardinals $\leq \gamma$ below α_1 .

If S is a matrix on a carrier α and its prejumpt cardinal α_χ^\downarrow after χ conserves these cardinals, then we also say that S on α conserves these cardinals below α_1 .

2) We denote through $\mathbf{K}_{n+1}^\exists(\chi, \delta, \gamma, \alpha, \rho, S)$ the Π_{n-2} -formula:

$$\begin{aligned} & \sigma(\chi, \alpha, S) \wedge Lj^{< \alpha}(\chi) \wedge \chi < \delta < \gamma < \alpha \wedge S \triangleleft \rho \leq \chi^+ \wedge \rho = \widehat{\rho} \wedge \\ & \wedge SIN_n^{< \alpha_\chi^\downarrow}(\delta) \wedge SIN_{n+1}^{< \alpha_\chi^\downarrow} [< \rho] (\delta). \end{aligned}$$

Here, remind, the Π_{n-2} -formula $\sigma(\chi, \alpha, S)$ means that S is the singular matrix on its carrier α reduced to the cardinal χ (see definition 5.7 [27]); δ is the disseminator for S on α with

the base ρ of the level $n + 1$ (definition 6.9 [27]); the upper indices $< \alpha_\chi^\downarrow$ mean the bounding of the formula quantors under consideration by the prejump cardinal α_χ^\downarrow (see also definitions 2.3, 5.9 [27]); $\hat{\rho}$ is the closure of ρ under the pair function; and $Lj^{<\alpha}(\chi)$ is the Δ_1 -property of the cardinal χ saturation below α (see definition 6.9 4) [27]):

$$\chi < \alpha \wedge SIN_{n-1}^{<\alpha}(\chi) \wedge \Sigma rng(\tilde{\mathbf{S}}_n^{sin \triangleleft \chi}) \in B_\chi \wedge \sup dom(\tilde{\mathbf{S}}_n^{sin \triangleleft \chi}) = \chi.$$

We denote through $\mathbf{K}^{<\alpha_1}(\chi, \delta, \gamma, \alpha, \rho, S)$ the formula:

$$\mathbf{K}_n^{\forall < \alpha_1}(\gamma, \alpha_\chi^\downarrow) \wedge \mathbf{K}_{n+1}^{\exists \triangleleft \alpha_1}(\chi, \delta, \gamma, \alpha, \rho, S) \wedge \alpha < \alpha_1.$$

3) If this formula is fulfilled by the constants $\chi, \delta, \gamma, \alpha, \rho, S, \alpha_1$, then we say that $\chi, \delta, \alpha, \rho, S$ are strongly admissible for γ below α_1 .

If some of them are fixed or meant by the context, then we say that the others are also strongly admissible for them (and for γ) below α_1 .

4) The matrix S is called strongly disseminator matrix or, briefly, δ -matrix strongly admissible on the carrier α for $\gamma = \gamma_\tau^{<\alpha_1}$ below α_1 , iff it possesses some disseminator $\delta < \gamma$ with a base ρ strongly admissible for them (also below α_1).

In every case of this kind δ -matrix is denoted by the common symbol δS or S .

If $\alpha_1 = k$, or α_1 is pointed out by the context, then the upper indices $< \alpha_1, \triangleleft \alpha_1$ here and other mentionings about α_1 are dropped.

—

Further up to the end of § 7 the notions of admissibility and of δ -matrices will be considered to be *strongly* notions, so the term

“strongly” will be omitted. All matrices will be considered to be δ -matrices; as the reducing cardinal χ in what follows will be used the complete cardinal χ^* (see definition 5.4 [27]) – if the context will not indicate some other case.

Here one should pay attention also to the notion of the cardinal χ saturation below α , that is to the Δ_1 -property $Lj^{<\alpha}(\chi)$; from lemma 5.5 [27] it follows, that χ^* is the cardinal saturated below any $\alpha > \chi^*$, $\alpha \in SIN_{n-2}$.

The symbol χ^* in notations and formula writings will be often omitted for some shortening.

Further every bounding cardinal α_1 will belong to SIN_{n-2} and hold the condition

$$\chi^* < \alpha_1 \leq k \wedge A_n^{\triangleleft \alpha_1}(\chi^*) = \|u_n^{\triangleleft \alpha_1}(l)\|, \quad (7.1)$$

or $\alpha_1 = k$ (unless otherwise is specified by the context).

The cardinal $\alpha_1 \leq k$ here with this property will be called *equinformative* (equally informative) with the cardinal χ^* .

This term is introduced here because of the phenomenon: no Σ_n -proposition $\varphi(l)$ has *jump ordinals* after χ^* below α_1 (see definition 2.4 [27]). It is not hard to see, that it is equivalent to the following: for every generic extension $\mathfrak{M}[l]$ every Π_n -proposition $\varphi(l)$ which holds in $\mathfrak{M}[l]$ below χ^* , thereafter holds in this extension below α_1 due to (7.1) and $\alpha_1 \in \Pi_{n-2}$; thus every Π_n -proposition $\varphi(l)$ holds or not in both cases simultaneously for every generic extension $\mathfrak{M}[l]$ (see also comment 7 [27] to illustrate the importance of this notion).

One should pay attention to the important example of such cardinal: the prejump cardinal $\alpha_{\chi^*}^\downarrow$ after χ^* of every matrix carrier $\alpha > \chi^*$. Besides, it will be always assumed for χ^* and α_1 that

$$\forall \gamma < \alpha_1 \exists \gamma' \in [\gamma, \alpha_1[\ SIN_n^{<\alpha_1}(\gamma') \wedge cf(\alpha_1) \geq \chi^{*+}$$

for convenience of some formula transformations.

The boundaries $< \alpha_1$, $\triangleleft \alpha_1$ will be omitted, as usual, if $\alpha_1 = k$,

or α_1 is meant by the context.

Definition 7.2

Let $\chi^* < \alpha_1$.

1) We call as the matrix δ -function of the level n below α_1 reduced to χ^* the function

$$\delta S_f^{<\alpha_1} = (\delta S_\tau^{<\alpha_1})_\tau$$

taking the value for τ :

$$\delta S_\tau^{<\alpha_1} = \min_{\leq} \{S \triangleleft \chi^{*+} : \exists \delta, \alpha, \rho < \gamma_{\tau+1}^{<\alpha_1} \mathbf{K}^{<\alpha_1}(\delta, \gamma_\tau^{<\alpha_1}, \alpha, \rho, S)\};$$

2) the following accompanying ordinal functions are defined below α_1 :

$$\check{\delta}_f^{<\alpha_1} = (\check{\delta}_\tau^{<\alpha_1})_\tau; \quad \rho_f^{<\alpha_1} = (\rho_\tau^{<\alpha_1})_\tau; \quad \alpha_f^{<\alpha_1} = (\alpha_\tau^{<\alpha_1})_\tau$$

taking the values:

$$\check{\delta}_\tau^{<\alpha_1} = \min \{\delta < \gamma_\tau^{<\alpha_1} : \exists \alpha, \rho < \gamma_{\tau+1}^{<\alpha_1} \mathbf{K}^{<\alpha_1}(\delta, \gamma_\tau^{<\alpha_1}, \alpha, \rho, \delta S_\tau^{<\alpha_1})\};$$

$$\rho_\tau^{<\alpha_1} = \min \{\rho < \chi^{*+} : \exists \alpha < \gamma_{\tau+1}^{<\alpha_1} \mathbf{K}^{<\alpha_1}(\check{\delta}_\tau^{<\alpha_1}, \gamma_\tau^{<\alpha_1}, \alpha, \rho, \delta S_\tau^{<\alpha_1})\};$$

$$\alpha_\tau^{<\alpha_1} = \min \{\alpha < \gamma_{\tau+1}^{<\alpha_1} : \mathbf{K}^{<\alpha_1}(\check{\delta}_\tau^{<\alpha_1}, \gamma_\tau^{<\alpha_1}, \alpha, \rho_\tau^{<\alpha_1}, \delta S_\tau^{<\alpha_1})\}.$$

For each matrix $\delta S_\tau^{<\alpha_1}$ these functions define its generating disseminator $\check{\delta}_\tau^{<\alpha_1} < \gamma_\tau^{<\alpha_1}$ along with its base $\rho_\tau^{<\alpha_1}$ and its carrier $\alpha_\tau^{<\alpha_1}$. \dashv

Using lemma 6.8 [27] it is easy to see, that here $\check{\delta}_\tau^{<\alpha_1}$ is the minimal with the base

$$\rho_\tau^{<\alpha_1} = \hat{\rho}_1, \quad \rho_1 = Od(\delta S_\tau^{<\alpha_1}),$$

that is the closure of the ordinal $Od(\delta S_\tau^{<\alpha_1})$ under the pair function; thereafter such disseminator is called the *generating eigendis-seminator* of $\delta S_\tau^{<\alpha_1}$ on $\alpha_\tau^{<\alpha_1}$ below α_1 and is denoted through $\check{\delta}_\tau^{<\alpha_1}$ (see also definition 6.9 2) [27]), and its base $\rho_\tau^{<\alpha_1}$ is denoted through $\rho_\tau^{<\alpha_1}$.

It is easy to obtain the following lemmas from these definitions and lemmas 5.15, 5.16 [27]:

Lemma 7.3

For $\alpha_1 < k$ the formulas $\mathbf{K}_n^{\forall < \alpha_1}$, $\mathbf{K}^{<\alpha_1}$ belong to Δ_1 and therefore all functions

$$\delta S_f^{<\alpha_1}, \quad \check{\delta}_f^{<\alpha_1}, \quad \rho_f^{<\alpha_1}, \quad \alpha_f^{<\alpha_1}$$

are Δ_1 -definable through χ^*, α_1 .

For $\alpha_1 = k$ the formulas \mathbf{K}_n^\forall , \mathbf{K} belong to Σ_n and these functions are Δ_{n+1} -definable. \dashv

Lemma 7.4 (About δ -function absoluteness)

Let $\chi^* < \gamma_{\tau+1}^{<\alpha_1} < \alpha_2 < \alpha_1 \leq k$, $\alpha_2 \in SIN_{n-2}^{<\alpha_1}$ and

$$(\gamma_\tau^{<\alpha_1} + 1) \cap SIN_n^{<\alpha_2} = (\gamma_\tau^{<\alpha_1} + 1) \cap SIN_n^{<\alpha_1},$$

then on the set

$$\{\tau' : \chi^* \leq \gamma_{\tau'}^{<\alpha_2} \leq \gamma_\tau^{<\alpha_1}\}$$

the functions

$$\delta S_f^{<\alpha_2}, \quad \check{\delta}_f^{<\alpha_2}, \quad \rho_f^{<\alpha_2}, \quad \alpha_f^{<\alpha_2}$$

coincide respectively with the functions

$$\delta S_f^{<\alpha_1}, \quad \check{\delta}_f^{<\alpha_1}, \quad \rho_f^{<\alpha_1}, \quad \alpha_f^{<\alpha_1}.$$

\dashv

The following lemma and the reasoning proving it present the idea which will be applied further in various significant typical situations:

Lemma 7.5 (About disseminator)

1) *Let*

$$(i) \quad]\tau_1, \tau_2[\subseteq \text{dom}(\delta S_f^{<\alpha_1}) , \quad \gamma_{\tau_2} \in SIN_n^{<\alpha_1};$$

$$(ii) \quad \tau_3 \in \text{dom}(\delta S_f^{<\alpha_1}) , \quad \tau_2 \leq \tau_3;$$

$$(iii) \quad \check{\delta}_{\tau_3}^{<\alpha_1} < \gamma_{\tau_2}^{<\alpha_1} .$$

Then

$$\check{\delta}_{\tau_3}^{<\alpha_1} \leq \gamma_{\tau_1}^{<\alpha_1} .$$

2) *Let δ -matrix S on its carrier α be admissible for $\gamma_\tau^{<\alpha_1}$ along with its disseminator δ and base ρ below α_1 , then:*

$$(i) \quad \{\tau' : \delta < \gamma_{\tau'}^{<\alpha_1} \leq \gamma_\tau^{<\alpha_1}\} \subseteq \text{dom}(\delta S_f^{<\alpha_1});$$

(ii) *this matrix S along with the same δ, ρ possesses the minimal admissible carrier $\alpha' \in]\gamma_\tau^{<\alpha_1}, \gamma_{\tau+1}^{<\alpha_1}[$.*

Proof. 1) The upper indices $< \alpha_1, \triangleleft \alpha_1$ will be dropped. Let us consider the matrix $S^3 = \delta S_{\tau_3}$ and $\check{\delta}^3 = \check{\delta}_{\tau_3}$, $\rho^3 = \rho_{\tau_3}$. Suppose 1) fails, then by (iii)

$$\gamma_{\tau_1} < \check{\delta}^3 < \gamma_{\tau_2} \quad \text{and} \quad \check{\delta}^3 = \gamma_{\tau_4}$$

for some $\tau_4 \in]\tau_1, \tau_2[$. Let us observe the situation below, standing on $\alpha^3 = \alpha_{\tau_3}^{\downarrow}$. From (i) and lemma 7.4 it comes that

$$\delta S_f^{<\alpha^3} \equiv \delta S_f \quad \text{on} \quad]\tau_1, \tau_2[$$

and the matrix $S^4 = \delta S_{\tau_4}^{<\alpha^3} = \delta S_{\tau_4}$ on the carrier $\alpha_{\tau_4}^{<\alpha^3} = \alpha_{\tau_4}$ has the disseminator

$$\check{\delta}^4 = \check{\delta}_{\tau_4}^{<\alpha^3} = \check{\delta}_{\tau_4} < \gamma_{\tau_4} = \check{\delta}^3 \text{ with the base } \rho^4 = \rho_{\tau_4}^{<\alpha^3}.$$

Now the argument from the proof of lemma 6.6 [27] should be repeated. From $\check{\delta}^4 < \check{\delta}^3$ it comes that

$$\rho^4 < \rho^3 \text{ and that is why } \check{\delta}^4 \notin SIN_{n+1}^{<\alpha^3}[< \rho_{\tau_3}]$$

and by lemma 6.6 [27] (for $m = n + 1$) there exists some Σ_n -proposition $\varphi(\alpha, \vec{a})$ with the train \vec{a} of constants $< \rho_{\tau_3}$ and some ordinal $\alpha_0 \in [\check{\delta}^4, \alpha^3[$ such that

$$\forall \alpha < \alpha_0 \quad \varphi^{<\alpha^3}(\alpha, \vec{a}) \wedge \neg \varphi^{<\alpha^3}(\alpha_0, \vec{a}).$$

The disseminator $\check{\delta}^3$ restricts the proposition $\exists \alpha \neg \varphi(\alpha, \vec{a})$ below α^3 , so $\alpha_0 \in]\check{\delta}^4, \check{\delta}^3[$. The Π_{n+1} -proposition

$$\begin{aligned} \forall \alpha, \gamma \Big(\neg \varphi(\alpha, \vec{a}) \longrightarrow \exists \gamma_1 (\gamma < \gamma_1 \wedge SIN_{n-1}(\gamma_1) \wedge \\ \wedge \exists \delta < \alpha \quad \exists \alpha', \mathbf{K}(\delta, \gamma_1, \alpha', \rho^4, S^4)) \Big) \end{aligned}$$

is fulfilled below $\check{\delta}^3$ and hence $\check{\delta}^3$ extends it up to α^3 , because

$$S^4 \triangleleft \rho_4 < \rho_3.$$

Hence, for every $\gamma_{\tau}^{<\alpha^3} > \check{\delta}^3$ there appears δ -matrix S^4 admissible on some carrier

$$\alpha \in [\gamma_{\tau}^{<\alpha^3}, \alpha^3[\text{ for } \gamma_{\tau}^{<\alpha^3}$$

along with its disseminator $\check{\delta}^4 < \check{\delta}^3$ and the base ρ^4 .

From here it follows that below α^3 there are definable the *minimal*

cardinal $\check{\delta}^m$ and the *minimal* base ρ^m with this property, that is fulfilling the following statement *below* α^3 :

$$\exists \gamma^m \forall \gamma > \gamma^m (SIN_{n-1}(\gamma) \rightarrow \exists \alpha', S \mathbf{K}(\check{\delta}^m, \gamma, \alpha', \rho^m, S)),$$

that is there exists $\gamma^m < \alpha^3$ such that for every $\gamma_\tau^{\leq \alpha^3} \in]\gamma^m, \alpha^3[$ there exists some δ -matrix S admissible on some carrier $\alpha \in [\gamma_\tau^{\leq \alpha^3}, \alpha^3[$ for $\gamma_\tau^{\leq \alpha^3}$ below α^3 along with its generating disseminator $\check{\delta}^m < \gamma^m$ with the base ρ^m .

Obviously, $\check{\delta}^m < \check{\delta}^3$. Since the minimal value ρ^m is definable below α^3 , by lemma 4.6 [27] about spectrum type, it follows

$$\rho^m < OT(\delta S_{\tau_3}) \triangleleft Od(\delta S_{\tau_3}).$$

But then it implies the contradiction: there exist δ -matrix S^m on some carrier $\alpha^m \in]\gamma_{\tau_3}, \alpha^3[$ admissible for γ_{τ_3} along with the disseminator $\check{\delta}^m < \gamma_{\tau_3}$ and the base ρ^m and by condition \mathbf{K}_{n+1}^\exists

$$S^m \triangleleft \rho^m < OT(\delta S_{\tau_3}) \leq Od(\delta S_{\tau_3}),$$

though δS_{τ_3} is \leq -minimal by definition 7.2.

Statement 2) (i) repeats lemma 5.17 2) (i) [27] and follows from definition 7.2 immediately; while statement 2) (ii) one can establish easily by means of the argument of lemma 5.17 2) (ii) [27] proof for the matrix S instead of $S_{\chi_\tau}^{\leq \alpha^1}$ and for the formula \mathbf{K} instead of σ ; we shall return to this argument in § 8 in the more important case. \dashv

The unrelativized function δS_f really does exist on the final subinterval of the subinaccessible cardinal k as it shows

Lemma 7.6 (About δ -function definiteness)

There exists an ordinal $\delta < k$ such that δS_f is defined on the set

$$T = \{\tau : \delta < \gamma_\tau < k\}.$$

The minimal of such ordinals δ is denoted by δ^* , its successor in SIN_n by δ^{*1} and the following corresponding ordinals are introduced:

$$\tau_1^* = \tau(\delta^*), \quad \tau^{*1} = \tau(\delta^{*1}),$$

$$\text{so that } \delta^* = \gamma_{\tau_1^*}, \quad \delta^{*1} = \gamma_{\tau^{*1}},$$

$$\text{and } \alpha^{*1} = \alpha_{\tau^{*1}}^\downarrow, \quad \rho^{*1} = \rho_{\tau^{*1}}.$$

Proof consists in the immediate application of lemma 6.14 [27] for $\alpha_1 = k$, $m = n + 1$, $\chi = \chi^*$. \dashv

Lemma 7.7

$$\delta^* \in SIN_n \cap SIN_{n+1}^{<\alpha^{*1}} [< \rho^{*1}]^{'}$$

Proof. Let us consider the disseminator $\check{\delta}_{\tau^{*1}}$ with the base ρ^{*1} of the matrix $\delta S_{\tau^{*1}}$ on the carrier $\alpha_{\tau^{*1}}$. Since

$$\delta^{*1} \in SIN_n, \quad \check{\delta}_{\tau^{*1}} < \delta^{*1}$$

and

$$\check{\delta}_{\tau^{*1}} \in SIN_n^{<\alpha^{*1}} \cap SIN_{n+1}^{<\alpha^{*1}} [< \rho^{*1}],$$

lemma 3.8 [27] implies $\check{\delta}_{\tau^{*1}} \in SIN_n$ and by lemmas 7.5 2), 7.6 $\check{\delta}_{\tau^{*1}} = \delta^*$. \dashv

Definition 7.8

1. The function $\delta S_{\tau}^{<\alpha_1}$ is called *monotone on an interval* $[\tau_1, \tau_2[$ or on corresponding interval $[\gamma_{\tau_1}^{<\alpha_1}, \gamma_{\tau_2}^{<\alpha_1}[$ below α_1 , iff $\tau_1 + 1 < \tau_2$, $]\tau_1, \tau_2[\subseteq \text{dom}(\alpha S_f^{<\alpha_1})$ and

$$\forall \tau' \tau'' (\tau_1 < \tau' < \tau'' < \tau_2 \longrightarrow \delta S_{\tau'}^{<\alpha_1} \leq \delta S_{\tau''}^{<\alpha_1}).$$

2. Thereafter the function δS_f is called (totally) *monotone* iff for $\tau_1^* = \tau(\delta^*)$:

$$\forall \tau' \tau'' (\tau_1^* < \tau' < \tau'' < k \longrightarrow \delta S_{\tau'} \leq \delta S_{\tau''}).$$

—

Some easy fragments of the matrix function δS_f monotonicity comes from definition 7.2 and lemma 7.5 2) (ii) at once:

Lemma 7.9 (About δ -function monotonicity)

Let

$$\tau_1 < \tau_2 \text{ and } \check{\delta}_{\tau_2}^{<\alpha_1} < \gamma_{\tau_1}^{<\alpha_1}.$$

Then

$$\delta S_{\tau_1}^{<\alpha_1} \leq \delta S_{\tau_2}^{<\alpha_1}.$$

—

Let us discuss the situation which arises.

We have revealed above, that the simplest matrix function S_f is \leq -monotone, but for every $\tau > \tau^*$ the prejump cardinals α^{\downarrow} of S_{τ} on its corresponding carrier $\alpha \in]\gamma_{\tau}, \gamma_{\tau+1}[$ do not conserve the subinaccessibility of levels $\geq n$ of cardinals $\leq \gamma_{\tau}$, and some other important properties of the lower levels of the universe are destroyed when relativizing to α^{\downarrow} (see lemmas 5.17, 5.18 and their discussion in the end of §5 [27]).

In order to overcome this obstruction we have supplied the values of this function, matrices S_τ , by disseminators of the level $n+1$ and required the conservation of the subinaccessibility of the level n for cardinals $\leq \gamma_\tau$, that is we passed to the δ -function δS_f .

But now it involves the new complication: now with the help of lemmas 7.3-7.7 above one can see, that after this modification the δ -function is deprived of the property of total \leq -monotonicity on $[\tau_1^*, k[$, and just due to the fact that in many cases the prejump cardinals α^\Downarrow of δ -matrices carriers α , vice versa, *give rise to the subinaccessibility of the level n* of some cardinals $\leq \gamma_\tau$ that become subinaccessible (relatively to α^\Downarrow), not being those in the universe (Kiselev [13]).

The way out of this new situation is pointed out by the following discovery that affords the solution of the problem:

One can see that the matrix δS_{τ_0} breaking the \leq -monotonicity on $[\tau_1^*, k[$ at the first time, that is for

$$\tau_0 = \sup\{\tau : \delta S_f \text{ is } \leq\text{-monotone on }]\tau_1^*, \tau[\},$$

is placed on some carrier $\alpha_{\tau_0} \in]\delta^*, \delta^{*1}[$ and also $\delta S_{\tau_0} \triangleleft \rho^{*1}$ by lemma 3.2 [27].

Therefore from lemmas 7.7, 6.3 [27] (for $m = n+1$, $\alpha_1 = \alpha^{*1}$) it follows that the disseminator $\delta_{\tau^{*1}}$ carries over precisely the same situation, but below $\alpha^0 = \alpha_{\tau_0}^\Downarrow$, that is:

the class $SIN_n^{<\alpha^0}$ contains some cardinals $\gamma_{\tau_1}^{<\alpha^0} < \gamma_{\tau_2}^{<\alpha^0}$ such that

$$]\tau_1, \tau_2[\subseteq \text{dom}(\delta S_f^{<\alpha^0})$$

and again just the same matrix

$$\delta S_{\tau^0} = \delta S_{\tau_0'}^{<\alpha^0}$$

is breaking the monotonicity of $\delta S_f^{<\alpha^0}$ on $] \tau_1, \tau_2[$ for the first time for some ordinal $\tau'_0 \in] \tau_1, \tau_2[$, but below α^0 .

So, here we come to the third and final approach to the main idea:

The following requirements should be imposed on δ -matrices:

1) *they must possess the property of “autoexorcizivity”, that is of self-exclusion in such situations of monotonicity violation; the matrices with this property (of “unit characteristic”) will have the priority over other matrices (of “zero characteristic” respectively) during defining of the matrix function;*

2) *one more requirement should be imposed on the matrices of zero characteristic, hampering their forming: their disseminator data bases must increase substantially, when the proceeding part of matrix function, that is have already been defined, contains monotonicity violation, in order to correct this fault – the using of matrices of zero characteristic;*

on this grounds the δ -matrix function should receive inconsistent properties of monotonicity and nonmonotonicity simultaneously.

Obviously, all these reasons require the recursive definition of the matrix function, setting its values depending on the properties of its preceding values.

We start to realize this idea from the following section.

8 Matrix α -Functions

For the forthcoming recursive definition it is necessary to complicate the previous formula \mathbf{K}_{n+1}^\exists (definition 7.1). But beforehand certain subformulas are to be introduced in view to the more clearness of the construction of this formula, where the variable X_1 plays the role of the matrix function $\alpha S_f^{<\alpha}$ and the variable X_2 plays the role of the characteristic function $a_f^{<\alpha}$ forthcoming to be defined below α both; the latter function assigns corresponding characteristics (unit or zero) to reduced matrices serving as values of $\alpha S_f^{<\alpha}$; these characteristics of matrices on their carriers will take values unit $a = 1$ or zero $a = 0$ according to the principle sketched above.

During introducing, these formulas will be accompanied by comments on their sense, and after resulting definition 8.2 we shall describe in outline how it works as a whole.

All these formulas were used in the author's previous works [10–17], but some of them were scattered over the text in their certain forms (sometimes nonformalized, some others in semantic manner), and here they are gathered together; also some suitable redesignations are used.

In these formulas various cardinals from the classes SIN_n , SIN_{n-1} , SIN_{n-2} , of subinaccessibility are used. It is necessary to take in view that after $<-$ or $<\lhd$ -bounding of these formulas by some cardinal α (see definition 2.3 [27]) there arise the subinaccessibility classes of the same level, but bounded by this α ; for example the SIN_n -subinaccessibility turns into the $SIN_n^{<\alpha}$ -subinaccessibility, but below α ; thus all formulas after that narrate about corresponding situation below α .

Such transformations lean on definitions and on lemmas 3.3-3.8 [27].

Definition 8.1

The following auxiliary formulas are introduced:

I. Intervals of matrix function definiteness

1.0 $A_0(\chi, \tau_1, \tau_2, X_1)$:

$$\begin{aligned} & \tau_1 + 1 < \tau_2 \wedge (X_1 \text{ is a function on }]\tau_1, \tau_2[) \wedge \\ & \wedge \tau_1 = \min \{ \tau :]\tau, \tau_2[\subseteq \text{dom}(X_1) \} \wedge \\ & \wedge \exists \gamma^1 (\chi \leq \gamma^1 = \gamma_{\tau_1} \wedge \text{SIN}_n(\gamma^1)); \end{aligned}$$

so, this formula means, that the interval $[\tau_1, \tau_2[$ takes up the special place in relation to the matrix function X_1 domain: this function is defined on $] \tau_1, \tau_2[$ and τ_1 is the minimal ordinal with this property; besides that the cardinal γ_{τ_1} belongs to SIN_n ; due to this minimality X_1 always is not defined for such ordinal τ_1 .

1.1 $A_1(\chi, \tau_1, \tau_2, X_1)$:

$$A_0(\chi, \tau_1, \tau_2, X_1) \wedge \exists \gamma^2 (\gamma^2 = \gamma_{\tau_2} \wedge \text{SIN}_n(\gamma^2));$$

such interval $[\tau_1, \tau_2[$ and the corresponding interval $[\gamma_{\tau_1}, \gamma_{\tau_2}[$ will be called the intervals of the function X_1 definiteness maximal to the left (in $\text{dom}(X_1)$), maximal in the sense that there is no interval $] \tau', \tau_2[$ in $\text{dom}(X_1)$ with the lesser left end $\tau' < \tau_1$; in addition it is still demanded that $\gamma_{\tau_1} \in \text{SIN}_n$.

1.2 $A_{1.1}^M(\chi, \tau_1, \tau_2, X_1)$:

$$A_1(\chi, \tau_1, \tau_2, X_1) \wedge \tau_2 = \sup \{ \tau : A_1(\chi, \tau_1, \tau_2, X_1) \};$$

here the interval $] \tau_1, \tau_2[$ is the maximal (included in $\text{dom}(X_1)$), maximal in the sense that it is not included in any other interval

$] \tau'_1, \tau'_2[\subseteq \text{dom}(X_1)$ such that $\gamma_{\tau'_2} \in \text{SIN}_n$; beyond this condition it is still demanded that $\gamma_{\tau_1} \in \text{SIN}_n$; thus such interval $[\tau_1, \tau_2[$, and the corresponding interval $[\gamma_{\tau_1}, \gamma_{\tau_2}[$ will be called the maximal intervals of the function X_1 definiteness.

1.3 $A_{1,2}(\tau_1, \tau_2, \eta)$:

$$\exists \gamma^1, \gamma^2 \left(\gamma^1 = \gamma_{\tau_1} \wedge \gamma^2 = \gamma_{\tau_2} \wedge \right. \\ \left. \wedge \eta = OT(\{\gamma : \gamma^1 < \gamma < \gamma^2 \wedge \text{SIN}_n(\gamma)\}) \right);$$

here, remind, OT denote the order type of the specified set, therefore we shall call such ordinal η the type of the interval $[\tau_1, \tau_2[$ and also of the corresponding interval $[\gamma_{\tau_1}, \gamma_{\tau_2}[$.

1.4 $A_2(\chi, \tau_1, \tau_2, \tau_3, X_1)$:

$$A_1(\chi, \tau_1, \tau_3, X_1) \wedge \tau_1 + 1 < \tau_2 < \tau_3 \wedge \tau_2 = \\ = \sup \left\{ \tau < \tau_3 : \forall \tau', \tau'' (\tau_1 < \tau' < \tau'' < \tau \longrightarrow X(\tau') \leq X(\tau'')) \right\};$$

so, here τ_2 is the minimal index at which the \leq -monotonicity of the matrix function X_1 fails.

1.5 $A_3(\chi, \tau_1, \tau'_1, \tau_2, \tau_3, X_1, X_2)$:

$$A_2(\chi, \tau_1, \tau_2, \tau_3, X_1) \wedge \tau_1 < \tau'_1 < \tau_2 \wedge (X_2 \text{ is a function on }]\tau_1, \tau_3[) \wedge \\ \wedge \tau'_1 = \min \{ \tau \in]\tau_1, \tau_2[: X_1(\tau) \geq X_1(\tau_2) \wedge X_2(\tau) = 1 \};$$

thus, here is indicated that the \leq -monotonicity of the matrix function X_1 on $] \tau_1, \tau_3[$ is broken first at the index τ_2 and just

because of the matrix $X_1(\tau'_1) \succ X_1(\tau_2)$ for $\tau'_1 \in]\tau_1, \tau_2[$ of unit characteristic.

$$1.6.a \quad A_4^b(\chi, \tau_1, \tau'_1, \tau_2, \tau_3, \eta, X_1, X_2):$$

$$A_3(\chi, \tau_1, \tau'_1, \tau_2, \tau_3, X_1, X_2) \wedge A_{1.2}(\tau_1, \tau_3, \eta);$$

$$1.6.a(i) \quad A_4^b(\chi, \tau_1, \tau_2, \eta, X_1, X_2):$$

$$\exists \tau'_1, \tau'_2 \leq \tau_2 \quad A_4^b(\chi, \tau_1, \tau'_1, \tau'_2, \tau_2, \eta, X_1, X_2);$$

$$1.6.b \quad A_4^{Mb}(\chi, \tau_1, \tau'_1, \tau_2, \tau_3, \eta, X_1, X_2):$$

$$A_4^b(\chi, \tau_1, \tau'_1, \tau_2, \tau_3, \eta, X_1, X_2) \wedge A_{1.1}^M(\chi, \tau_1, \tau_3, X_1);$$

$$1.6.b(i) \quad A_4^{Mb}(\chi, \tau_1, \tau_2, \eta, X_1, X_2):$$

$$\exists \tau'_1, \tau'_2 \leq \tau_2 \quad A_4^{Mb}(\chi, \tau_1, \tau'_1, \tau'_2, \tau_2, \eta, X_1, X_2);$$

in what follows every interval $[\tau_1, \tau_3[$ possessing this property A_4^b for some τ'_1, τ_2, η and the corresponding interval $[\gamma_{\tau_1}, \gamma_{\tau_3}[$ will be called the blocks of the type η , and if there in addition holds $A_{1.1}^M(\chi, \tau_1, \tau_3, X_1)$ – then the maximal blocks.

Such blocks are considered further as objectionable because of there fatal defect: the violating of monotonicity. Because of that we shall impose on such blocks certain hard conditions in order to avoid their formation in the course of matrix function recursive defining (see the condition \mathbf{K}^0 below).

Let us hold up for a little while this definition 8.1 to explain the sense and direction of its subsequent part.

All formulas and notions introduced above and also forthcoming will be used in the resulting recursive definition 8.2 in bounded forms, that is their variables and constants will be $<-$ or $<|$ -bounded by some corresponding cardinal α_1 . In such cases their

present formulations are used, but with the added remark “below α_1 ”; respectively their designations are supplied by the upper index $< \alpha_1$ or $\triangleleft \alpha_1$.

Thereby

$$A_1^{\triangleleft \alpha_1}(\chi, \tau_1, \tau_2, X_1)$$

is the formula:

$$\begin{aligned} & \tau_1 + 1 < \tau_2 \wedge (X_1 \text{ is a function on }]\tau_1, \tau_2[) \wedge \\ & \wedge \tau_1 = \min \{ \tau :]\tau, \tau_2[\subseteq \text{dom}(X_1) \} \wedge \\ & \wedge \exists \gamma^1, \gamma^2 (\chi \leq \gamma^1 \wedge \gamma^1 = \gamma_{\tau_1}^{< \alpha_1} \wedge \gamma^2 = \gamma_{\tau_2}^{< \alpha_1} \wedge \\ & \wedge \text{SIN}_n^{< \alpha_1}(\gamma^1) \wedge \text{SIN}_n^{< \alpha_1}(\gamma^2)), \end{aligned}$$

which means that $] \tau_1, \tau_2[$ is the interval from domain of the function X_1 with the minimal left end τ_1 , and in addition the corresponding cardinals $\gamma_{\tau_1}^{< \alpha_1}, \gamma_{\tau_2}^{< \alpha_1}$ are $\text{SIN}_n^{< \alpha_1}$ -cardinals – all it below α_1 .

Respectively,

$$A_4^{b \triangleleft \alpha_1}(\chi, \tau_1, \tau'_1, \tau_2, \tau_3, \eta, X_1, X_2)$$

is the formula:

$$A_3^{\triangleleft \alpha_1}(\chi, \tau_1, \tau'_1, \tau_2, \tau_3, X_1, X_2) \wedge A_{1.2}^{\triangleleft \alpha_1}(\tau_1, \tau_3, \eta)$$

which means that $[\tau_1, \tau_3[$ and $[\gamma_{\tau_1}^{< \alpha_1}, \gamma_{\tau_3}^{< \alpha_1}[$ are the *blocks* below α_1 of type η , that is the interval $] \tau_1, \tau_3[$ is maximal to the left included in $\text{dom}(X_1)$ and the cardinals $\gamma_{\tau_1}^{< \alpha_1}, \gamma_{\tau_3}^{< \alpha_1}$ are in $\text{SIN}_n^{< \alpha_1}$ both, and \leq -monotonicity of X_1 on $] \tau_1, \tau_3[$ is broken first at the index $\tau_2 \in] \tau_1, \tau_3[$ and just because of the matrix $X_1(\tau'_1) \succ X_1(\tau_2)$ of *unit* characteristic for some $\tau'_1 \in] \tau_1, \tau_2[$ – and all it below α_1 .

Is is not hard to see that all these and forthcoming bounded formulas under consideration belong to the class Δ_1^1 for any $\alpha_1 > \chi$, $\alpha_1 < k$, $\alpha_1 \in SIN_{n-2}$.

To introduce the notions forthcoming clearly it is convenient to clarify in outline the principle regulating the assignment of characteristics to matrices on their carriers and intersection of these characteristics with each other, because the characteristic function will play the leading role in recursive definition 8.2 of matrix function.

So, the matrix S on its carrier α — and this carrier α itself — will receive *zero* characteristic $a = 0$, if it participate in violation of matrix function monotonicity in the following sense:
there exist some interval of the matrix function definiteness

$$[\gamma_{\tau_1}^{<\alpha_\chi^\downarrow}, \gamma_{\tau_3}^{<\alpha_\chi^\downarrow}[$$

below the prejump cardinal α_χ^\downarrow after χ of this carrier α , where occurs the *same matrix* S as the value of the matrix function X_1 , but already below α_χ^\downarrow :

$$X_1(\tau_2) = S,$$

for the index $\tau_2 \in]\tau_1, \tau_3[$ which is the minimal one violating the monotonicity of X_1 below α_χ^\downarrow , that is when there holds

$$A_2^{<\alpha_\chi^\downarrow}(\chi, \tau_1, \tau_2, \tau_3, X_1).$$

And here comes the last refinement of this notion: in addition there must be no admissible matrices for γ_{τ_1} below α_χ^\downarrow and all values of the matrix function X_1 on the interval $] \tau_1, \tau_2]$ must be of *unit* characteristic:

$$\forall \tau (\tau_1 < \tau \leq \tau_2 \rightarrow X_2(\tau) = 1). \quad 9)$$

Otherwise S on α and α itself will receive *unit* characteristic $a = 1$.

And while the matrix function will receive its recursive definition 8.2 forthcoming, matrices of unit characteristic will receive the priority over matrices of zero characteristic — to avoid the violation of monotonicity of this function.

It is natural to realize the notion of “priority” in the sense: when some value $X_1(\tau)$ of matrix function X_1 is on definition and there are matrices S^0, S^1 of characteristic zero and unit respectively that can be nominated as such value, then just matrix S^1 should be assigned as $X_1(\tau)$.

But there will be certain rather specific cases, when zero characteristic will be rejected by certain other conditions, when *zero matrix* S on its carrier α will be forbidden for nomination for a value of matrix functions; in every such case we shall say, that S on α is *suppressed*.

We use here the term “suppression”, not “nonpriority”, because such suppression will happen not every time, but in special cases depending on disposition of the carrier α .

So, now we come to the description of cases, when such suppression takes place.

To organize these cases in the proper way and to formulate the suppression condition one should notice, that formulas above in this current definition 8.1 must be used in the following special way:

Till now in all these formulas 1.0–1.6 b (i) above the symbols X_1, X_2 were treated as functions defined on ordinals.

But for recursive definition 8.2 of matrix functions it is necessary to use functions defined on *pairs* of ordinals. Therefore let us introduce for such function X another function

$$X[\alpha] = \{(\tau, \eta) : ((\alpha, \tau), \eta) \in X\},$$

so that

$$X(\alpha, \tau) = X[\alpha](\tau)$$

for every pair $(\alpha, \tau) \in \text{dom}(X)$.

Thereafter these formulas in definition 8.2 and formulas forthcoming will be used often for X_1, X_2 as functions:

$$X_1[\alpha^0], X_2[\alpha^0],$$

where α^0 are some ordinals.

Now let us return to definition 8.1 with the aim to form the so called suppression condition; it arises in connection with coverings of cardinals by blocks of special kind and for this aim the following band of conditions is needed:

II. Suppression conditions

2.1a. $A_{5.1}^{sc}(\chi, \gamma^m, \gamma, X_1, X_2)$:

$$\gamma^m < \gamma \wedge \forall \gamma' \in [\gamma^m, \gamma[\exists \tau_1, \tau_2, \eta \left(\gamma^m \leq \gamma_{\tau_1} \leq \gamma' < \gamma_{\tau_2} \leq \gamma \wedge \right.$$

$$\left. \wedge A_4^{Mb}(\chi, \tau_1, \tau_2, \eta, X_1, X_2) \right) \wedge$$

$$\wedge \forall \gamma^{m'} < \gamma^m \neg \forall \gamma'' \in [\gamma^{m'}, \gamma[\exists \tau'_1, \tau'_2, \eta' \left(\gamma^{m'} \leq \gamma_{\tau'_1} \leq \gamma'' < \gamma_{\tau'_2} \leq \gamma \wedge \right.$$

$$\left. \wedge A_4^{Mb}(\chi, \tau'_1, \tau'_2, \eta', X_1, X_2) \right);$$

here is indicated, that the interval $[\gamma^m, \gamma[$ with the right end γ is the union of maximal blocks and that its left end γ^m is the minimal one with this property; such collection of intervals will be called the covering of the cardinal γ ; it is easy to see, that under this condition γ^m, γ are SIN_n -cardinals;

if one withdraw here the right end γ it cause the following condition:

2.1b. $A_{5.1}^{sc}(\chi, \gamma^m, X_1, X_2):$

$$\begin{aligned} & \forall \gamma' \geq \gamma^m \exists \tau_1, \tau_2, \eta \left(\gamma^m \leq \gamma_{\tau_1} \leq \gamma' < \gamma_{\tau_2} \wedge \right. \\ & \quad \left. \wedge A_4^{Mb}(\chi, \tau_1, \tau_2, \eta, X_1, X_2) \right) \wedge \\ & \wedge \forall \gamma^{m'} < \gamma^m \neg \forall \gamma'' \geq \gamma^{m'} \exists \tau'_1, \tau'_2, \eta' \left(\gamma^{m'} \leq \gamma_{\tau'_1} \leq \gamma'' < \gamma_{\tau'_2} \wedge \right. \\ & \quad \left. \wedge A_4^{Mb}(\chi, \tau'_1, \tau'_2, \eta', X_1, X_2) \right). \end{aligned}$$

In view to compose the suppression condition in a proper way the following special conditions should be superimposed on such coverings for the ordinals $\gamma^m < \gamma^ < \gamma, \eta^*$:*

2.2. $A_{5.2}^{sc}(\chi, \gamma^m, \gamma^*, \eta^*, X_1, X_2):$

$$\begin{aligned} & A_{5.1}^{sc}(\chi, \gamma^m, \gamma^*, X_1, X_2) \wedge \\ & \wedge \forall \tau_1, \tau_2, \eta \left(\gamma_{\tau_1} < \gamma_{\tau_2} \leq \gamma^* \wedge A_4^{Mb}(\chi, \tau_1, \tau_2, \eta, X_1, X_2) \rightarrow \eta < \eta^* \right) \wedge \\ & \quad \wedge \forall \eta < \eta^* \exists \gamma' < \gamma^* \forall \tau'_1, \tau'_2, \eta' \left(\gamma' < \gamma_{\tau'_2} \leq \gamma^* \wedge \right. \\ & \quad \left. \wedge A_4^{Mb}(\chi, \tau'_1, \tau'_2, \eta', X_1, X_2) \rightarrow \eta < \eta' \right); \end{aligned}$$

in this case, when $A_{5.2}^{sc}$ holds, we shall say, that covering types of the cardinal γ^ are nondecreasing up to η^* substantially; thereby the ordinal η^* is limit;*

2.3. $A_{5.3}^{sc}(\chi, \gamma^*, \gamma^1, \gamma, \eta^*, X_1, X_2):$

$$\begin{aligned} & \exists \tau_1, \tau \left(\gamma_{\tau_1} = \gamma^1 \wedge \gamma_{\tau} = \gamma \wedge A_4^b(\chi, \tau_1, \tau, \eta^*, X_1, X_2) \wedge \right. \\ & \quad \left. \forall \tau'_1, \tau'_2, \eta' \left(\gamma^* < \gamma_{\tau'_2} \leq \gamma^1 \wedge A_4^{Mb}(\chi, \tau'_1, \tau'_2, \eta', X_1, X_2) \rightarrow \right. \right. \\ & \quad \left. \left. \rightarrow \eta' = \eta^* \right) \right); \end{aligned}$$

now these three conditions should be assembled in the following

2.4. Suppressing covering condition

$$A_{5.4}^{sc}(\chi, \gamma, \eta^*, X_1, X_2):$$

$$\exists \gamma^m, \gamma^*, \gamma^1 \left(\gamma^m < \gamma^* < \gamma^1 < \gamma \wedge \eta^* < \chi^+ \wedge \right.$$

$$A_{5.1}^{sc}(\chi, \gamma^m, \gamma^1, X_1, X_2) \wedge A_{5.2}^{sc}(\chi, \gamma^m, \gamma^*, \eta^*, X_1, X_2) \wedge \\ \left. \wedge A_{5.3}^{sc}(\chi, \gamma^*, \gamma^1, \gamma, \eta^*, X_1, X_2) \right);$$

let us call the covering of γ possessing this property the suppressing covering for γ of the type η^* ;

so, these three conditions $A_{5.1}^{sc} - A_{5.3}^{sc}$ mean together, that the covering of the cardinal γ splits into three parts: its types are nondecreasing up to the ordinal $\eta^* < \chi^+$ substantially to the left of γ^* , and then stabilizes from γ^* up to γ^1 , that is the interval $[\gamma^*, \gamma^1[$ is covered by the maximal blocks of the constant type η^* , also there is the block $[\gamma^1, \gamma[$ of the same type $\eta^* < \chi^+$; it is clear that these conditions define the ordinals $\gamma^m < \gamma^* < \gamma^1 < \gamma$, η^* uniquely through γ (if they exist);

$$2.5. \quad A_{5.5}^{sc}(\chi, \gamma, \eta^*, \alpha, X_1, X_2):$$

$$\forall \gamma' \left(\gamma \leq \gamma' < \alpha \rightarrow \exists \tau'_1, \tau'_2, \eta' \left(\gamma_{\tau'_1}^{<\alpha} \leq \gamma' < \gamma_{\tau'_2}^{<\alpha} \wedge \right. \right.$$

$$\left. \wedge A_4^{Mb < \alpha}(\chi, \tau'_1, \tau'_2, \eta', X_1, X_2) \wedge \eta' \geq \eta^* \right);$$

here is indicated, that the interval $[\gamma, \alpha[$ is covered by maximal blocks below α of types $\eta' \geq \eta^*$;

Now all these conditions should be composed in the following integrated

2.6. Resulting suppressing condition

$$A_5^{S,0}(\chi, a, \gamma, \alpha, \rho, S, X_1^0, X_2^0, X_1, X_2):$$

$$\begin{aligned} & a = 0 \wedge SIN_n(\gamma) \wedge \rho < \chi^+ \wedge \sigma(\chi, \alpha, S) \wedge \\ & \wedge \exists \eta^*, \tau < \gamma \left(\gamma = \gamma_\tau \wedge A_{5.4}^{sc}(\chi, \gamma, \eta^*, X_1^0, X_2^0) \wedge \right. \\ & \quad \wedge \forall \tau' (\tau < \tau' \wedge SIN_n(\gamma_{\tau'}) \rightarrow \\ & \rightarrow \exists \alpha', S' [\gamma_{\tau'} < \alpha' < \gamma_{\tau'+1} \wedge SIN_n^{<\alpha'_{\chi}^{\downarrow}}(\gamma_{\tau'}) \wedge \sigma(\chi, \alpha', S') \wedge \\ & \quad \wedge A_{5.5}^{sc}(\chi, \gamma, \eta^*, \alpha'_{\chi}^{\downarrow}, X_1[\alpha'_{\chi}^{\downarrow}], X_2[\alpha'_{\chi}^{\downarrow}])) \Big); \end{aligned}$$

this last condition will superimpose on the matrix S on its carrier α the rather hard requirements and if it can be realized, then only in the very special cases:

the reduced matrix S must be of zero characteristic on its carrier α , the cardinal γ must be SIN_n -subinaccessible, the base ρ must be strictly less than χ^+ , the cardinal γ must be covered by the suppressing covering of the type η^* ; moreover, for all $\gamma' > \gamma$, $\gamma' \in SIN_n$ there exist carriers $\alpha' > \gamma'$ with prejump cardinals $\alpha'_{\chi}^{\downarrow}$ preserving SIN_n -cardinals $\leq \gamma'$ and with the interval $[\gamma, \alpha'_{\chi}^{\downarrow}]$ covered by the maximal blocks of types $\eta' \geq \eta^*$ below $\alpha'_{\chi}^{\downarrow}$.

Further such occurrences of the matrix S of zero characteristic on its corresponding carriers α will fail while definition of the matrix function will go on, and, so, we shall say that here zero matrix S on α is suppressed for γ .

Respectively, zero matrix S on α with disseminator δ and base ρ is nonsuppressed for γ if this condition fails; thus any matrix S on α is nonsuppressed if it is unit or has the base $\rho \geq \chi^+$

on α , or γ is not SIN_n -cardinal; so one should always have in view the corresponding cardinal γ .

Let us suspend this definition 8.1 one more time for a little bit to describe the direction of its final part operation.

To construct the forthcoming definition 8.2 of matrix function in the required way it should be managed by Π_{n-2} -formula

$$U_{n-2}(\mathbf{n}, x, \chi, a, \delta, \gamma, \alpha, \rho, S)$$

which is universal for the formula class Π_{n-2} with denoted free variables

$$x, \chi, a, \delta, \gamma, \alpha, \rho, S,$$

and variable Gödel number \mathbf{n} of such formulas in basic model \mathfrak{M} (see Tarski [25], also Addison [23]).

When this number \mathbf{n} and the variable x will take certain special value \mathbf{n}^α simultaneously:

$$\mathbf{n} = x = \mathbf{n}^\alpha$$

then this formula along with the Σ_n -formula $\mathbf{K}_n^\forall(\gamma, \alpha_\chi^\downarrow)$ will state, that S is the α -matrix reduced to χ on the carrier α of characteristic a with its disseminator δ and the base ρ , *admissible* for γ and obeying certain complex recursive conditions; remind, the formula $\mathbf{K}_n^\forall(\gamma, \alpha)$ (definition 7.1) means, that the ordinal α preserves all SIN_n -cardinals $\leq \gamma$.

Nevertheless, until the value \mathbf{n}^α will be assigned to the variables \mathbf{n} , x , this formula will work in this definition 8.2 with $\mathbf{n} = x$:

$$U_{n-2}(x, x, \chi, a, \delta, \gamma, \alpha, \rho, S).$$

Also further the following function restrictions are used:

$$X|\tau^0 = \{(\tau, \eta) \in X : \tau < \tau^0\};$$

$$X|^1\alpha^0 = \{((\alpha, \tau), \eta) \in X : \alpha < \alpha^0\}.$$

Now we return to definition 8.1 for the last time. The suppression condition $A_5^{S,0}$ will operate in the following conjunction with the formula U_{n-2} , bearing all the definition of matrix functions forthcoming:

III. Bearing conditions

$$3.1 \quad U_{n-2}^*(\mathbf{n}, x, \chi, a, \delta, \gamma, \alpha, \rho, S, X_1^0|\tau', X_2^0|\tau', X_1|^1\alpha^0, X_2|^1\alpha^0) :$$

$$U_{n-2}(\mathbf{n}, x, \chi, a, \delta, \gamma, \alpha, \rho, S) \wedge \\ \wedge \neg A_5^{S,0}(\chi, a, \gamma, \alpha, \rho, S, X_1^0|\tau', X_2^0|\tau', X_1|^1\alpha^0, X_2|^1\alpha^0);$$

this condition along with $\mathbf{K}_n^\forall(\gamma, \alpha_\chi^\Downarrow)$ after their \triangleleft -bounding by the cardinal α^0 and for the constants

$$x = \mathbf{n}^\alpha, \chi, \delta, \gamma, \alpha, \rho, \tau' < \alpha^0, S \triangleleft \rho$$

will describe the following situation below α^0 : S is the matrix reduced to χ on its carrier α of the characteristic a admissible for γ along with its disseminator δ and the base ρ , which is nonsuppressed for γ below α^0 — and one should point out that this situation for any pair (α^0, τ') will be determined by the functions

$$X_1^0|\tau' = X_1[\alpha^0]|\tau', \quad X_2^0|\tau' = X_2[\alpha^0]|\tau', \quad \text{and} \quad X_1[\alpha_\chi'^\Downarrow], \quad X_2[\alpha_\chi'^\Downarrow]$$

for various $\alpha_\chi'^\Downarrow < \alpha^0$; therefore the recursion mode provided by this condition will work correctly.

$$3.2 \quad A^0(x, \chi, \tau):$$

$$\exists \gamma \left(\gamma = \gamma_\tau \wedge \neg \exists a, \delta, \alpha, \rho, S (\mathbf{K}_n^\forall(\gamma, \alpha_\chi^\Downarrow) \wedge \right. \\ \left. U_{n-2}(x, x, a, \delta, \gamma, \alpha, \rho, S) \right);$$

this condition for $x = \mathbf{n}^\alpha$ will mean, that there is no α -matrix S on some carrier α , admissible for γ_τ .

3.3 $A_2^0(x, \chi, \tau_1, \tau_2, \tau_3, X_1)$:

$$A^0(x, \chi, \tau_1) \wedge A_2(\chi, \tau_1, \tau_2, \tau_3, X_1).$$

IV. Closing condition

This condition will help to close the diagonal reasoning providing the final contradiction:

$$4.1 \quad \left(a = 0 \longrightarrow \forall \tau'_1, \tau''_1, \tau'_2, \tau'_3, \eta' < \alpha_\chi^\Downarrow [\gamma_{\tau'_1}^{<\alpha_\chi^\Downarrow} \leq \delta < \gamma_{\tau'_3}^{<\alpha_\chi^\Downarrow} \wedge \right. \\ \left. \wedge A_4^{Mb \triangleleft \alpha_\chi^\Downarrow}(\chi, \tau'_1, \tau''_1, \tau'_2, \tau'_3, \eta', X_1[\alpha_\chi^\Downarrow], X_2[\alpha_\chi^\Downarrow]) \rightarrow \eta' < \rho \vee \rho = \chi^+ \right];$$

this formula has the following content for any zero matrix S on the carrier α with the disseminator δ and base ρ :

if this disseminator falls in any maximal block $[\gamma_{\tau'_1}^{<\alpha_\chi^\Downarrow}, \gamma_{\tau'_3}^{<\alpha_\chi^\Downarrow}[$ below the prejumpt cardinal α_χ^\Downarrow , that is if

$$\gamma_{\tau'_1}^{<\alpha_\chi^\Downarrow} \leq \delta < \gamma_{\tau'_3}^{<\alpha_\chi^\Downarrow},$$

then this base ρ has to arise substantially and to exceed the type η' of this very block, or even to take its value the greatest possible:

$$\eta' < \rho \vee \rho = \chi^+,$$

for lack of anything better;

therefore in similar cases the interval $[\tau'_1, \tau'_3[$ and the corresponding interval

$$[\gamma_{\tau'_1}^{<\alpha_\chi^\Downarrow}, \gamma_{\tau'_3}^{<\alpha_\chi^\Downarrow}[$$

will be severe for this zero matrix S on α and will hamper the using of S on α (with this disseminator δ).

V. *Equinformative condition*5.1 $A_6^e(\chi, \alpha^0)$:

$$\chi < \alpha^0 \wedge A_n^{\leq \alpha^0}(\chi) = \|u_n^{\leq \alpha^0}(\underline{l})\| \wedge SIN_{n-2}(\alpha^0) \wedge \\ \wedge \forall \gamma < \alpha^0 \exists \gamma_1 \in [\gamma, \alpha^0[\quad SIN_n^{\leq \alpha^0}(\gamma_1);$$

the cardinal α^0 here with this property is called, remind, equinformative with χ .

—

The latter notion was used above several times (see also [11–17], [27]) and here it is emphasized because of its special importance: every Π_n -proposition $\varphi(l)$ holds or not in any generic extension $\mathfrak{M}[l]$ below χ and also in this extension below α^0 simultaneously (see comment after (7.1)); the best example of such α^0 – any prejump cardinal α_χ^\downarrow after χ of any matrix carrier $\alpha > \chi$ (if this cardinal is limit for the class $SIN_n^{\leq \alpha_\chi^\downarrow}$).

Now everything is ready to assemble all the fragments introduced above in the following *integrated* definition 8.2 where the variable matrix S on its carrier α is under consideration.

Requirements which are superimposed there on S on α and on its disseminator δ with the data base ρ depend on the functions X_i , $i = \overline{1, 5}$, that already are recursively defined below the prejump cardinal α_χ^\downarrow ; they are defined on the certain subset of the domain

$$\mathcal{A}_\chi^{\alpha_\chi^\downarrow} = \left\{ (\alpha^0, \tau) : \exists \gamma < \alpha^0 (\chi < \gamma = \gamma_\tau^{\leq \alpha^0} \wedge \right. \\ \left. \wedge \alpha^0 \leq \alpha_\chi^\downarrow \wedge A_6^e(\chi, \alpha^0)) \right\} \quad (8.1)$$

and therefore the functions

$$X_i^0 = X_i[\alpha^0], \quad i = \overline{1, 5}$$

are defined on the corresponding subset of

$$\{\tau : \gamma_\tau \in SIN_{n-1}^{<\alpha^0}\}$$

for every cardinal $\alpha^0 \leq \alpha_\chi^\Downarrow$ equinformative with χ . This set $\mathcal{A}_\chi^{\alpha_\chi^\Downarrow}$ is considered to be lexicographically ordered (with α^0 as the first component in this order and τ as the second).

So, the variable X_2^0 will play here the role of characteristic function $a_f^{<\alpha^0}$ defined below the cardinal α^0 ; X_1^0 will play the role of matrix function $\alpha S_f^{<\alpha^0}$; X_3^0 – the role of disseminator function $\tilde{\delta}_f^{<\alpha^0}$; X_4^0 – the role of its data base function $\rho_f^{<\alpha^0}$; X_5^0 – the role of the carrier function $\alpha_f^{<\alpha^0}$; all of them will be defined below α^0 .

After all these functions will be defined for all such cardinals

$$\alpha^0 \leq \alpha_\chi^\Downarrow$$

then in conclusion the resulting requirement will be superimposed on the matrix S on its carrier α under consideration along with its disseminator δ and the data base ρ depending on location of this δ , more precisely – depending on the maximal block

$$[\gamma_{\tau'_1}^{<\alpha_\chi^\Downarrow}, \gamma_{\tau'_3}^{<\alpha_\chi^\Downarrow}[$$

containing this disseminator δ , that is already been defined below α_χ^\Downarrow .

And here, remind, is the closing requirement superimposed on the matrix S on its carrier α , mentioned above:

if S on α has zero characteristic and its admissible disseminator δ falls in the maximal block of type η' below α_χ^\Downarrow , then $\eta' < \rho \wedge \rho = \chi^+$; thus, in such case data base ρ has to increase considerably and we shall see that it is possible, but every time leads to some contradiction.

In addition here is needed the formulation $Lj^{<\alpha}(\chi)$ of the saturated cardinal χ notion (see argument before (7.1) or definition 6.9 4) [27]); remind also that $\widehat{\rho}$ denote the closure of ρ under the pair function.

So, the recursive definition, based on the set $\mathcal{A}_\chi^{\alpha_\chi^\Downarrow}$, starts: ¹⁰⁾

Definition 8.2

1) Let

$$U_{n-2}(\mathbf{n}, x, \chi, a, \delta, \gamma, \alpha, \rho, S)$$

be the Π_{n-2} -formula universal for the class Π_{n-2} where \mathbf{n} is variable Gödel number of Π_{n-2} -formulas with the free variables $x, \chi, a, \delta, \gamma, \alpha, \rho, S$, and let

$$U_{n-2}^*(x, \chi, a, \delta, \gamma, \alpha, \rho, S, X_1^0|\tau', X_2^0|\tau', X_1^1|\alpha^0, X_2^1|\alpha^0).$$

be the formula

$$U_{n-2}(x, x, \chi, a, \delta, \gamma, \alpha, \rho, S) \wedge$$

$$\wedge \neg A_5^{S,0}(\chi, a, \gamma, \alpha, \rho, S, X_1^0|\tau', X_2^0|\tau', X_1^1|\alpha^0, X_2^1|\alpha^0).$$

2) Let

$$A_7^{RC}(x, \chi, X_1, X_2, X_3, X_4, X_5, \alpha_\chi^\Downarrow)$$

be the following Δ_1 -formula, providing the required recursive condition:

$$\begin{aligned}
& \bigwedge_{1 \leq i \leq 5} \left((X_i \text{ is a function}) \wedge X_i \triangleleft \alpha_\chi^{\Downarrow+} \wedge \right. \\
& \left. \wedge \text{dom}(X_i) \subseteq \left\{ (\alpha^0, \tau) : \exists \gamma < \alpha^0 (\chi < \gamma = \gamma_\tau^{\leq \alpha^0} \wedge \right. \right. \\
& \left. \left. \wedge \alpha^0 \leq \alpha_\chi^{\Downarrow} \wedge A_6^e(\chi, \alpha^0)) \right\} \right) \wedge \\
& \wedge \forall \alpha^0 \left((\alpha^0 \leq \alpha_\chi^{\Downarrow} \wedge A_6^e(\chi, \alpha^0)) \longrightarrow \right. \\
& \longrightarrow \exists X_1^0, X_2^0, X_3^0, X_4^0, X_5^0, X_1^{1,0}, X_2^{1,0} \left[\bigwedge_{1 \leq i \leq 5} X_i^0 = X_i[\alpha^0] \wedge \right. \\
& \left. \wedge X_1^{1,0} = X_1|{}^1 \alpha^0 \wedge X_2^{1,0} = X_2|{}^1 \alpha^0 \wedge \right. \\
& \left. \wedge \forall \tau', \gamma', \gamma'' < \alpha^0 \left(\chi < \gamma' \wedge \gamma' = \gamma_{\tau'}^{\leq \alpha^0} \wedge \gamma'' = \gamma_{\tau'+1}^{\leq \alpha^0} \longrightarrow \right. \right. \\
& \left. \forall a' \left(X_2^0(\tau') = a' \leftrightarrow \right. \right. \\
& \left. \left. \leftrightarrow a' = \max_{\leq} \left\{ a'' : \exists \delta'', \alpha'', \rho'' < \gamma'' \exists S'' \triangleleft \chi^+ (\mathbf{K}_n^{\forall < \alpha^0}(\gamma', \alpha_\chi^{\Downarrow}) \wedge \right. \right. \right. \\
& \left. \left. \wedge U_{n-2}^{* \leq \alpha^0}(x, \chi, a'', \delta'', \gamma', \alpha'', \rho'', S'', X_1^0|{}^1 \tau', X_2^0|{}^1 \tau', X_1^{1,0}, X_2^{1,0})) \right\} \right) \wedge \\
& \left. \wedge \forall S' \left(X_1^0(\tau') = S' \longleftrightarrow \exists a' \left(a' = X_2^0(\tau') \wedge \right. \right. \right. \\
& \left. \left. \wedge S' = \min_{\leq} \left\{ S'' \triangleleft \chi^+ : \exists \delta'', \alpha'', \rho'' < \gamma'' (\mathbf{K}_n^{\forall < \alpha^0}(\gamma', \alpha_\chi^{\Downarrow}) \wedge \right. \right. \right. \\
& \left. \left. \wedge U_{n-2}^{* \leq \alpha^0}(x, \chi, a', \delta'', \gamma', \alpha'', \rho'', S'', X_1^0|{}^1 \tau', X_2^0|{}^1 \tau', X_1^{1,0}, X_2^{1,0})) \right\} \right) \wedge
\end{aligned}$$

$$\begin{aligned}
& \wedge \forall \delta' \left(X_3^0(\tau') = \delta' \longleftrightarrow \exists a', S' \left(a' = X_1^0(\tau') \wedge S' = X_2^0(\tau') \wedge \right. \right. \\
& \quad \left. \wedge \delta' = \min_{\leqslant} \left\{ \delta'' < \gamma' : \exists \alpha'', \rho'' < \gamma'' (\mathbf{K}_n^{\forall < \alpha^0}(\gamma', \alpha_X^{\prime\downarrow}) \wedge \right. \right. \\
& \quad \left. \left. \wedge U_{n-2}^{*\triangleleft \alpha^0}(x, \chi, a', \delta'', \gamma', \alpha'', \rho'', S', X_1^0|\tau', X_2^0|\tau', X_1^{1,0}, X_2^{1,0})) \right\} \right) \Bigg) \wedge \\
& \quad \wedge \forall \rho' \left(X_4^0(\tau') = \rho' \longleftrightarrow \exists a', S', \delta' \left(a' = X_1^0(\tau') \wedge S' = X_2^0(\tau') \wedge \right. \right. \\
& \quad \left. \wedge \delta' = X_3^0(\tau') \wedge \rho' = \min_{\leqslant} \left\{ \rho'' < \chi^+ : \exists \alpha'' < \gamma'' (\mathbf{K}_n^{\forall < \alpha^0}(\gamma', \alpha_X^{\prime\downarrow}) \wedge \right. \right. \\
& \quad \left. \left. \wedge U_{n-2}^{*\triangleleft \alpha^0}(x, \chi, a', \delta', \gamma', \alpha'', \rho'', S', X_1^0|\tau', X_2^0|\tau', X_1^{1,0}, X_2^{1,0})) \right\} \right) \Bigg) \wedge \\
& \quad \wedge \forall \alpha' \left(X_5^0(\tau') = \alpha' \longleftrightarrow \exists a', S', \delta', \rho' \left(a' = X_1^0(\tau') \wedge S' = X_2^0(\tau') \wedge \right. \right. \\
& \quad \left. \delta' = X_3^0(\tau') \wedge \rho' = X_4^0(\tau') \wedge \alpha' = \min_{\leqslant} \left\{ \alpha'' < \gamma'' : \mathbf{K}_n^{\forall < \alpha^0}(\gamma', \alpha_X^{\prime\downarrow}) \wedge \right. \right. \\
& \quad \left. \left. \wedge U_{n-2}^{*\triangleleft \alpha^0}(x, \chi, a', \delta', \gamma', \alpha'', \rho', S', X_1^0|\tau', X_2^0|\tau', X_1^{1,0}, X_2^{1,0})) \right\} \right) \Bigg) \Bigg].
\end{aligned}$$

3) We denote through

$$\alpha \mathbf{K}_{n+1}^{\Xi}(x, \chi, a, \delta, \gamma, \alpha, \rho, S)$$

the Π_{n-2} -formula which is equivalent to the following formula:

$$(a = 0 \vee a = 1) \wedge \sigma(\chi, \alpha, S) \wedge Lj^{<\alpha}(\chi) \wedge \chi < \delta < \gamma < \alpha \wedge \\ \wedge S \triangleleft \rho \leq \chi^+ \wedge \rho = \hat{\rho} \wedge SIN_n^{<\alpha \downarrow}(\delta) \wedge SIN_{n+1}^{<\alpha \downarrow}[\prec \rho](\delta) \wedge$$

$$\begin{aligned}
& \wedge \forall \gamma < \alpha_\chi^\Downarrow \exists \gamma' \in [\gamma, \alpha_\chi^\Downarrow[\text{SIN}_n^{<\alpha_\chi^\Downarrow}(\gamma') \wedge cf(\alpha_\chi^\Downarrow) \geq \chi^+ \wedge \\
& \wedge \exists X_1, X_2, X_3, X_4, X_5 \left\{ A_7^{RC}(x, \chi, X_1, X_2, X_3, X_4, X_5, \alpha_\chi^\Downarrow) \wedge \right. \\
& \left(a = 0 \longleftrightarrow \exists \tau'_1, \tau'_2, \tau'_3 < \alpha_\chi^\Downarrow \left(A_2^{0\triangleleft\alpha_\chi^\Downarrow}(\chi, \tau'_1, \tau'_2, \tau'_3, X_1[\alpha_\chi^\Downarrow]) \wedge \right. \right. \\
& \left. \left. \wedge \forall \tau'' (\tau'_1 < \tau'' \leq \tau'_2 \rightarrow X_2[\alpha_\chi^\Downarrow](\tau'') = 1) \wedge X_1[\alpha_\chi^\Downarrow](\tau'_2) = S \right) \wedge \right. \\
& \left. \wedge \left(a = 0 \longrightarrow \forall \tau'_1, \tau''_1, \tau'_2, \tau'_3, \eta' < \alpha_\chi^\Downarrow \left[\gamma_{\tau'_1}^{<\alpha_\chi^\Downarrow} \leq \delta < \gamma_{\tau'_3}^{<\alpha_\chi^\Downarrow} \wedge \right. \right. \\
& \left. \wedge A_4^{Mb\triangleleft\alpha_\chi^\Downarrow}(\chi, \tau'_1, \tau''_1, \tau'_2, \tau'_3, \eta', X_1[\alpha_\chi^\Downarrow], X_2[\alpha_\chi^\Downarrow]) \rightarrow \right. \\
& \left. \left. \rightarrow \eta' < \rho \vee \rho = \chi^+ \right] \right) \Big\}.
\end{aligned}$$

Let us denote by $\mathbf{K}^0(\chi, a, \delta, \alpha, \rho)$ the last conjunctive constituent in big curly brackets $\{ , \}$ in the latter formula, that is the closing condition:

$$\begin{aligned}
& \left(a = 0 \longrightarrow \forall \tau'_1, \tau''_1, \tau'_2, \tau'_3, \eta' < \alpha_\chi^\Downarrow \left[\gamma_{\tau'_1}^{<\alpha_\chi^\Downarrow} \leq \delta < \gamma_{\tau'_3}^{<\alpha_\chi^\Downarrow} \wedge \right. \right. \\
& \left. \wedge A_4^{Mb\triangleleft\alpha_\chi^\Downarrow}(\chi, \tau'_1, \tau''_1, \tau'_2, \tau'_3, \eta', X_1[\alpha_\chi^\Downarrow], X_2[\alpha_\chi^\Downarrow]) \rightarrow \right. \\
& \left. \left. \rightarrow \eta' < \rho \vee \rho = \chi^+ \right] \right).^{11)}
\end{aligned}$$

The functions $X_1[\alpha_\chi^\Downarrow], X_2[\alpha_\chi^\Downarrow]$ are not mentioned here in the denotation of \mathbf{K}^0 for brevity taking into account, that they are defined uniquely in the preceding part of this formula $\alpha\mathbf{K}_{n+1}^\exists$.

4) The formula $\alpha \mathbf{K}_{n+1}^\exists$ is the Π_{n-2} -formula and thereby it receives its Gödel number \mathbf{n}^α , that is:

$$\alpha \mathbf{K}_{n+1}^\exists(x, \chi, a, \delta, \gamma, \alpha, \rho, S) \longleftrightarrow U_{n-2}(\mathbf{n}^\alpha, x, \chi, a, \delta, \gamma, \alpha, \rho, S).$$

Let us assign the value \mathbf{n}^α to the variable x in this equivalence and everywhere further, thereafter these \mathbf{n}^α, x will be dropped in the notations.

We denote through $\alpha \mathbf{K}^{<\alpha_1}(\chi, a, \delta, \gamma, \alpha, \rho, S)$ the Δ_1 -formula:

$$\mathbf{K}_n^{\forall < \alpha_1}(\gamma, \alpha_\chi^\Downarrow) \wedge \alpha \mathbf{K}_{n+1}^{\exists \triangleleft \alpha_1}(\chi, a, \delta, \gamma, \alpha, \rho, S) \wedge \alpha < \alpha_1,$$

and, respectively, through $\alpha \mathbf{K}^{* < \alpha_1}(\chi, a, \delta, \gamma, \alpha, \rho, S)$ – the formula, which is obtained from the formula $\alpha \mathbf{K}^{<\alpha_1}$ through joining to it the conjunctive condition of the matrix S nonsuppression on α for γ (see definition 8.1 2.6), but below $\alpha_1 < k$ (as it was done above in points 2), 3) for α_χ^\Downarrow , but now for $\alpha_1 < k$ instead of α_χ^\Downarrow) – in the following way:

$$\begin{aligned} & \mathbf{K}_n^{\forall < \alpha_1}(\gamma, \alpha_\chi^\Downarrow) \wedge \alpha \mathbf{K}_{n+1}^{\exists \triangleleft \alpha_1}(\chi, a, \delta, \gamma, \alpha, \rho, S) \wedge \alpha < \alpha_1 \wedge \\ & \wedge \neg \left(a = 0 \wedge \text{SIN}_n^{<\alpha_1}(\gamma) \wedge \rho < \chi^+ \wedge \sigma(\chi, \alpha, S) \wedge \right. \\ & \wedge \exists X_1, X_2, X_3, X_4, X_5 \left(A_7^{RC}(\mathbf{n}^\alpha, \chi, X_1, X_2, X_3, X_4, X_5, \alpha_1) \wedge \right. \\ & \wedge \exists \eta^*, \tau < \gamma \left(\gamma = \gamma_\tau^{<\alpha_1} \wedge A_{5.4}^{sc \triangleleft \alpha_1}(\chi, \gamma, \eta^*, X_1[\alpha_1]|\tau, X_2[\alpha_1]|\tau) \wedge \right. \\ & \wedge \forall \tau' \left(\tau < \tau' \wedge \text{SIN}_n^{<\alpha_1}(\gamma_{\tau'}^{<\alpha_1}) \rightarrow \right. \\ & \exists \alpha', S' \left[\gamma_{\tau'}^{<\alpha_1} < \alpha' < \gamma_{\tau'+1}^{<\alpha_1} \wedge \text{SIN}_n^{<\alpha_\chi^\Downarrow}(\gamma_{\tau'}^{<\alpha_1}) \wedge \sigma(\chi, \alpha', S') \wedge \right. \\ & \left. \left. \left. \left. \left. \wedge A_{5.5}^{sc \triangleleft \alpha_1}(\chi, \gamma, \eta^*, \alpha_\chi'^\Downarrow, X_1[\alpha_\chi'^\Downarrow], X_2[\alpha_\chi'^\Downarrow]) \right) \right) \right) \right) \right) \right); \end{aligned}$$

here is stated the admissibility of S on α , and in addition — its nonsuppressibility for γ below α_1 . So, if $\alpha \mathbf{K}^{<\alpha_1}$ holds, but

$\alpha\mathbf{K}^{*<\alpha_1}$ fails, then S on α is admissible but suppressed (all it below α_1).¹²⁾

5) If the formula $\alpha\mathbf{K}^{<\alpha_1}(\chi, a, \delta, \gamma, \alpha, \rho, S)$ is fulfilled by the constants $\chi, a, \delta, \alpha, \gamma, \rho, S, \alpha_1$, then we say that $\chi, a, \delta, \alpha, \rho, S$ are admissible very strongly for γ below α_1 .

If some of them are fixed or meant by the context, then we say that others are admissible very strongly for them (and for γ) below α_1 .

So, by

$$\alpha\mathbf{K}^{<\alpha_1}(\chi, \gamma, \alpha, S)$$

will be denoted the formula

$$\exists a, \delta, \rho < \gamma \alpha\mathbf{K}^{<\alpha_1}(\chi, a, \delta, \gamma, \alpha, \rho, S)$$

meaning that S on α is admissible very strongly for γ below α_1 .

6) The matrix S is called autoexorcizive or, briefly, α -matrix admissible very strongly on the carrier α of the characteristic a for $\gamma = \gamma_\tau^{<\alpha_1}$ below α_1 , iff it possesses on α some disseminator $\delta < \gamma$ with a base ρ admissible very strongly for them also below α_1 .

In every case of this kind α -matrix is denoted by the general symbol αS or S .

If $a_1 = k$, or α_1 is pointed out by the context, then the upper indices $< \alpha_1, \triangleleft \alpha_1$ and other mentionings about α_1 are dropped.

Further all notions of admissibility will be always considered to be very strongly, so the term “very strongly” will be omitted in what follows. \dashv

Thus here the bounded formula

$$\alpha\mathbf{K}^{*<\alpha_1}(\chi, a, \delta, \gamma, \alpha, \rho, S)$$

arises from $\alpha\mathbf{K}^{<\alpha_1}$ by adding the condition of nonsuppression of the matrix S on α for γ below α_1 which is obtained

from the condition $\neg A_5^{S,0}$ in the way indicated above through its \triangleleft -bounding by the cardinal α_1 (that is through \triangleleft -bounding its individual variables by α_1 and through replacing its constituents $SIN_n(\gamma)$ with $SIN_n^{<\alpha_1}(\gamma)$).

So, any matrix S on α if *suppressed* for the cardinal γ below α_1 , then when γ is $SIN_n^{<\alpha_1}$ -cardinal, and when this S has zero characteristic on α and the base $\rho < \chi^+$ below α_1 .

Everywhere further $\chi = \chi^* < \alpha_1$; we shall often omit the notations of the functions $X_1 = \alpha S_f^{<\alpha_1}$, $X_2 = a_f^{<\alpha_1}$ and the symbols χ^* , \mathbf{n}^α in writings of all formulas

$$A_0 - A_5^{S,0}, \quad A^0, \quad A_2^0, \quad \alpha \mathbf{K}_{n+1}^\exists, \quad \mathbf{K}^0, \quad \alpha \mathbf{K}^{<\alpha_1}, \quad \alpha \mathbf{K}^{* < \alpha_1}$$

from definitions 8.1, 8.2 and of other denotations for some shortening (if it will not perform a misunderstanding); for example any prejump cardinals $\alpha_{\chi^*}^\downarrow$ will be denoted through α^\downarrow and so on. Concerning these formulas it should be pointed out, that definition 8.2 has been constructed with the aim to receive the key formula $\alpha \mathbf{K}_{n+1}^\exists$ of the class Π_{n-2} . To this end all constituent formulas were \triangleleft -bounded by ordinals α^0 or α^\downarrow .

But in what follows these boundaries will be often dropped without loss of their content, because their individual variables and constants are in fact will be bounded by ordinals pointed out in the context during their applications.

Clearly, variables X_i , $i = \overline{1,5}$ are defined in definition 8.2 uniquely through all their parameters, thus similar functions can be defined recursively following this construction by the same recursion on the similar set (remind the set (8.1))

$$\mathcal{A} = \{(\alpha_1, \tau) : \exists \gamma < \alpha^1(\chi^* < \gamma = \gamma_\tau^{<\alpha_1} \wedge A_6^e(\chi, \alpha_1))\}$$

of pairs (α_1, τ) (ordered lexicographically as in the former case, with α_1 as the first component in this order and τ as the second).

Definition 8.3

Let $\chi^* < \alpha_1$.

1) We call as the characteristic function of the level n below α_1 reduced to χ^* the function

$$a_f^{<\alpha_1} = (a_\tau^{<\alpha_1})_\tau$$

taking the values:

$$\begin{aligned} a_\tau^{<\alpha_1} &= \\ &= \max_{\leq} \{a : \exists \delta, \alpha, \rho < \gamma_{\tau+1}^{<\alpha_1} \exists S \triangleleft \chi^{*+} \alpha \mathbf{K}^{*<\alpha_1}(a, \delta, \gamma_\tau^{<\alpha_1}, \alpha, \rho, S)\}; \end{aligned}$$

2) we call as the matrix autoexorcizive (in monotonicity violation) function or, briefly, α -function of the level n below α_1 reduced to χ^* the function

$$\alpha S_f^{<\alpha_1} = (\alpha S_\tau^{<\alpha_1})_\tau$$

taking the values

$$\begin{aligned} \alpha S_\tau^{<\alpha_1} &= \\ &= \min_{\leq} \{S \triangleleft \chi^{*+} : \exists \delta, \alpha, \rho < \gamma_{\tau+1}^{<\alpha_1} \alpha \mathbf{K}^{*<\alpha_1}(a_\tau^{<\alpha_1}, \delta, \gamma_\tau^{<\alpha_1}, \alpha, \rho, S)\}; \end{aligned}$$

3) the following accompanying ordinal functions are defined below α_1

$$\begin{aligned} \text{the floating disseminator function} & \quad \tilde{\delta}_f^{<\alpha_1} = (\tilde{\delta}_\tau^{<\alpha_1})_\tau, \\ \text{its data base function} & \quad \rho_f^{<\alpha_1} = (\rho_\tau^{<\alpha_1})_\tau, \\ \text{the carrier function} & \quad \alpha_f^{<\alpha_1} = (\alpha_\tau^{<\alpha_1})_\tau, \\ \text{the generating disseminator function} & \quad \check{\delta}_f^{<\alpha_1} = (\check{\delta}_\tau^{<\alpha_1})_\tau, \end{aligned}$$

taking the values for $a_\tau = a_\tau^{<\alpha_1}$, $S_\tau = \alpha S_\tau^{<\alpha_1}$:

$$\begin{aligned} \tilde{\delta}_\tau^{<\alpha_1} &= \\ &= \min_{\leq} \{ \delta < \gamma_\tau^{<\alpha_1} : \exists \alpha, \rho < \gamma_{\tau+1}^{<\alpha_1} \alpha \mathbf{K}^{*<\alpha_1}(a_\tau, \delta, \gamma_\tau^{<\alpha_1}, \alpha, \rho, S_\tau) \}; \end{aligned}$$

$$\begin{aligned} \rho_\tau^{<\alpha_1} &= \\ &= \min_{\leq} \{ \rho \leq \chi^{*+} : \exists \alpha < \gamma_{\tau+1}^{<\alpha_1} \alpha \mathbf{K}^{*<\alpha_1}(a_\tau, \tilde{\delta}_\tau^{<\alpha_1}, \gamma_\tau^{<\alpha_1}, \alpha, \rho, S_\tau) \}; \end{aligned}$$

$$\begin{aligned} \alpha_\tau^{<\alpha_1} &= \\ &= \min_{\leq} \{ \alpha < \gamma_{\tau+1}^{<\alpha_1} : \alpha \mathbf{K}^{*<\alpha_1}(a_\tau, \tilde{\delta}_\tau^{<\alpha_1}, \gamma_\tau^{<\alpha_1}, \alpha, \rho_\tau^{<\alpha_1}, S_\tau) \}; \end{aligned}$$

and for $\alpha^1 = \alpha_\tau^{<\alpha_1^\downarrow}$:

$$\check{\delta}_\tau^{<\alpha_1} = \min_{\leq} \{ \delta < \gamma_\tau^{<\alpha_1} : SIN_n^{<\alpha_1}(\delta) \wedge SIN_{n+1}^{<\alpha_1} [<\rho_\tau^{<\alpha_1}] (\delta) \};$$

The value $a_\tau^{<\alpha_1}$ is called, remind, the characteristic of the matrix $\alpha S_\tau^{<\alpha_1}$ on the carrier $\alpha_\tau^{<\alpha_1}$, and of this carrier itself.

All the functions

$$a_f^{<\alpha_1}, \quad \tilde{\delta}_f^{<\alpha_1}, \quad \check{\delta}_f^{<\alpha_1}, \quad \rho_f^{<\alpha_1}$$

are called, for some brevity, the accessories of the functions

$$\alpha_f^{<\alpha_1}, \quad \alpha S_f^{<\alpha_1},$$

and their values for the index τ are called also the accessories of the values

$$\alpha_\tau^{<\alpha_1}, \quad \alpha S_\tau^{<\alpha_1};$$

similarly the function $\alpha_f^{<\alpha_1}$ is called the accessory of $\alpha S_f^{<\alpha_1}$, and its value $\alpha_\tau^{<\alpha_1}$ – the accessory of the matrix $\alpha S_\tau^{<\alpha_1}$ below α_1 , and so on.

–

The notion of characteristic is introduced in the general case:

Definition 8.4

We call as a characteristic of a matrix S on a carrier $\alpha > \chi^*$ the number $a(S, \alpha) = a$ defined in the following way:

$$\begin{aligned} & (a = 1 \vee a = 0) \wedge (a = 0 \longleftrightarrow \\ & \longleftrightarrow \exists \tau'_1, \tau'_2, \tau'_3 < \alpha^\Downarrow (A_2^{0 \triangleleft \alpha^\Downarrow}(\tau'_1, \tau'_2, \tau'_3, \alpha S_f^{<\alpha^\Downarrow}) \wedge \\ & \wedge \forall \tau'' (\tau'_1 < \tau'' \leq \tau'_2 \rightarrow a_{\tau''}^{<\alpha^\Downarrow} = 1) \wedge \alpha S_{\tau'_2}^{<\alpha^\Downarrow} = S)) \end{aligned}$$

The matrix S on its carrier α is called the unit matrix on α iff it has the unit characteristic on α ; otherwise it is called zero matrix on α .

–

Thereafter when the α -function $\alpha S_f^{<\alpha_1}$ is defined, the priority belongs to α -matrices possessing the greater characteristic.

This circumstance, although making possible the solution of the inaccessibility problem, complicates considerably the matrix function theory as a whole because the restriction reasoning does not work now freely: a situation concerning zero characteristic may not be carried over to the part of the universe below which is determined by unit characteristic, or by other reasons connected with suppressibility.

Definition 8.3 of α -function and accompanying ordinal functions follows the recursive definition 8.2 and since the functions X_i , $i = \overline{1, 5}$ are defined in the formula $\alpha\mathbf{K}_{n+1}^\exists$ uniquely through its parameters by this recursion, it is easy to see that functions $X_i[\alpha^0]$, $i = \overline{1, 5}$, in definition 8.2 coincide with corresponding functions

$$a_f^{<\alpha^0}, \quad \alpha S_f^{<\alpha^0}, \quad \tilde{\delta}_f^{<\alpha^0}, \quad \rho_f^{<\alpha^0}, \quad \alpha_f^{<\alpha^0}, \quad (8.2)$$

for every cardinal α^0 , equinformative with χ^* .

Cause of that we shall use their notations (8.2) instead of corresponding notations of these functions $X_i[\alpha^0]$, $i = \overline{1, 5}$ in formulas from definition 8.1 that is using these formulas but for the functions $X_i[\alpha^0]$, $i = \overline{1, 5}$, replaced with corresponding functions (8.2) for $\alpha^0 = \alpha_1$; we shall even omit them often for some brevity, when it will not cause misunderstanding and when the context will point out them clearly.

For instance, the formula $A_0^{<\alpha_1}(\tau_1, \tau_2, \alpha S_f^{<\alpha_1})$ means, that below α_1 there holds

$$\begin{aligned} & \tau_1 + 1 < \tau_2 \wedge (\alpha S_f^{<\alpha_1} \text{ is the function on }]\tau_1, \tau_2[) \wedge \\ & \wedge \tau_1 = \min\{\tau :]\tau, \tau_2[\subseteq \text{dom}(\alpha S_f^{<\alpha_1}) \wedge \\ & \wedge \chi^* \leq \gamma_{\tau_1}^{<\alpha_1} \wedge \gamma_{\tau_1}^{<\alpha_1} \in \text{SIN}_n^{<\alpha_1}; \end{aligned}$$

the formula $A_1^{<\alpha_1}(\tau_1, \tau_2, \alpha S_f^{<\alpha_1})$ means, that below α_1 there holds

$$A_0^{<\alpha_1}(\tau_1, \tau_2, \alpha S_f^{<\alpha_1}) \wedge \gamma_{\tau_2}^{<\alpha_1} \in \text{SIN}_n^{<\alpha_1};$$

the formula

$$A_2^{0<\alpha^\Downarrow}(\tau_1, \tau_2, \tau_3, \alpha S_f^{<\alpha^\Downarrow}) \wedge \forall \tau \in]\tau_1, \tau_2[\ a_\tau^{<\alpha^\Downarrow} = 1 \wedge \alpha S_{\tau_2}^{<\alpha^\Downarrow} = S$$

means that here α^\Downarrow is the prejump cardinal of α after χ^* , and there is no α -matrices admissible for $\gamma_{\tau_1}^{<\alpha_1}$ below α^\Downarrow , and below α^\Downarrow there holds

$$A_1^{<\alpha^\Downarrow}(\tau_1, \tau_3, \alpha S_f^{<\alpha^\Downarrow}),$$

where $\tau_2 \in]\tau_1, \tau_3[$ is the first ordinal at which monotonicity on $] \tau_1, \tau_3[$ of the matrix function $\alpha S_f^{<\alpha^\Downarrow}$ fails, but already below α^\Downarrow , and, moreover, $\alpha S_{\tau_2}^{<\alpha^\Downarrow} = S$ and all matrices $\alpha S_\tau^{<\alpha^\Downarrow}$ are of unit characteristic on $] \tau_1, \tau_2[$ – and so on.

Next, two easy remarks should be done:

1. All intervals $[\gamma_{\tau_1}^{<\alpha_1}, \gamma_{\tau_2}^{<\alpha_1}[$ of definiteness below α_1 , considered in definition 8.1 for the functions

$$X_1 = \alpha S_f^{<\alpha_1}, \quad X_2 = a_f^{<\alpha_1},$$

were of different types and were defined by different conditions, but all of them include the condition of the interval $[\gamma_{\tau_1}^{<\alpha_1}, \gamma_{\tau_2}^{<\alpha_1}[$ maximality to the left:

$$A_0^{<\alpha_1}(\tau_1, \tau_2, \alpha S_f^{<\alpha_1})$$

which states, among other things, that the matrix function $\alpha S_f^{<\alpha_1}$ below α_1 is defined on the interval $] \tau_1, \tau_2[$ and the ordinal τ_1 is the *minimal* one with this property and, moreover, $\gamma_{\tau_1}^{<\alpha_1}$ is the $SIN_n^{<\alpha_1}$ -cardinal. Due to this minimality it is not hard to see, that $\alpha S_f^{<\alpha_1}$ is not defined for this ordinal τ_1 itself!

2. The notions of admissibility, priority and nonsuppression should be distinguished. One can imagine two matrices S', S'' on their carriers α', α'' respectively along with their corresponding accessories, both admissible for one cardinal $\gamma_\tau^{<\alpha_1}$; when S' is of unit characteristic on α' it is always nonsuppressed and has the priority over S'' of zero characteristic on α'' . But even when there is no such matrix S' , still the matrix S'' on α'' can be suppressed, if there holds the suppression condition $A_5^{S,0}$ below α_1 ; in any case every matrix, being suppressed, can not be the value of the matrix function $\alpha S_f^{<\alpha_1}$.

So, for the interval $[\gamma_{\tau_1}^{<\alpha_1}, \gamma_{\tau_2}^{<\alpha_1}[$ maximal to the left below α_1 there can be no value $\alpha S_\tau^{<\alpha_1}$ for $\tau = \tau_1$, but still it does not exclude the existence of some matrix *only admissible* (but suppressed) for $\gamma_{\tau_1}^{<\alpha_1}$ below α_1 .

And now, with all these comments in hand, let us see how definition 8.2 – and, hence, definition 8.3 – works below α_1 (we consider, remind, the most important case when $\chi = \chi^*$, $\mathbf{n} = \mathbf{n}^\alpha$).

I. So, in the first part in the beginning of the formula

$$\alpha \mathbf{K}_{n+1}^{\exists < \alpha_1}(a, \delta, \gamma, \alpha, \rho, S)$$

it is stated, that S is the δ -matrix on its carrier $\alpha > \chi^*$, $\alpha < \alpha_1$ reduced to χ^* with the disseminator $\delta < \gamma$ and base ρ :

$$S \triangleleft \rho = \widehat{\rho} \leq \chi^{*+};$$

the prejump cardinal $\alpha^\Downarrow = \alpha_{\chi^*}^\Downarrow$ is limit for $SIN_n^{<\alpha^\Downarrow}$ and has the cofinality $\geq \chi^{*+}$; the disseminator δ has subinaccessibility below α^\Downarrow of the level n and even of the level $n+1$ with the base ρ , that is

$$\delta \in SIN_n^{<\alpha^\Downarrow} \cap SIN_{n+1}^{<\alpha^\Downarrow} [< \rho].$$

II. Then below α^\Downarrow there are defined the functions X_i , $i = \overline{1, 5}$ on pairs $(\alpha^0, \tau') \in \mathcal{A}_{\chi^*}^{\alpha^\Downarrow}$, where cardinals $\alpha^0 \in]\chi^*, \alpha^\Downarrow]$ are equinformative with χ^* and there exist cardinals $\gamma_{\tau'}^{<\alpha^0}$.

All these functions are recursively defined through definition of the functions X_i^0 , $i = \overline{1, 5}$, by means of the recursive condition A_7^{RC} :

$$X_1[\alpha^0] = \alpha S_f^{<\alpha^0}, \quad X_2[\alpha^0] = a_f^{<\alpha^0}, \quad X_3[\alpha^0] = \widetilde{\delta}_f^{<\alpha^0},$$

$$X_4[\alpha^0] = \rho_f^{<\alpha^0}, \quad X_5[\alpha^0] = \alpha_f^{<\alpha^0}.$$

The aim of this definition – to receive the resulting matrix function $\alpha S_f^{<\alpha^0}$, but the first it is introduced just the characteristic function

$$X_2^0 = X_2[\alpha^0] = a_f^{<\alpha^0}.$$

This function accepts the *maximal* possible values, unit or zero, that are characteristics of admissible matrices below α^0 , *but only not zero characteristic of suppressed zero matrices* S'' on their carriers α'' , that satisfies the suppression condition below α^0 :

$$A_5^{S,0 < \alpha^0}(0, \gamma_{\tau'}^{< \alpha^0}, \alpha'', \rho'', S'', X_1^0 | \tau', X_2^0 | \tau', X_1 |^1 \alpha^0, X_2 |^1 \alpha^0),$$

where the functions here

$$X_1^0 | \tau', \quad X_2^0 | \tau', \quad X_1 |^1 \alpha^0, \quad X_2 |^1 \alpha^0$$

are already defined. And everywhere further such suppressed zero matrices are systematically rejected.

After the characteristic function $X_2[\alpha^0] = a_f^{< \alpha^0}$ is defined, all remained functions

$$X_1[\alpha^0], \quad X_i[\alpha^0], \quad i = \overline{3, 5}$$

are defined one by one successively through the minimization of their admissible and nonsuppressed values.

So, the next it is defined the matrix function $X_1[\alpha^0] = \alpha S_f^{< \alpha^0}$, after that the corresponding disseminator function $X_3[\alpha^0] = \tilde{\delta}_f^{< \alpha^0}$, then the data base function $X_4[\alpha^0] = \rho_f^{< \alpha^0}$, and, in last turn, the carrier function $X_5[\alpha^0] = \alpha_f^{< \alpha^0}$ is defined.

The values of every subsequent of these functions depend essentially on the values of the previous ones.

III. After all these functions are constructed for every

$$\alpha^0 \in]\chi^*, \alpha^\Downarrow[,$$

the definition passes to the cardinal

$$\alpha^0 = \alpha^\Downarrow$$

and after that defines the characteristic of the matrix S on its carrier α itself:

S on α receives *zero characteristic*, if it participate in the following violation of the matrix function

$$X_1[\alpha^\Downarrow] = \alpha S_f^{<\alpha^\Downarrow}$$

monotonicity below α^\Downarrow , when below α^\Downarrow there holds the condition

$$\begin{aligned} \exists \tau'_1, \tau'_2, \tau'_3 < \alpha^\Downarrow (A_2^{0\triangleleft\alpha^\Downarrow}(\tau'_1, \tau'_2, \tau'_3, \alpha S_f^{<\alpha^\Downarrow}) \wedge \\ \wedge \forall \tau'' \in]\tau'_1, \tau'_2] a_{\tau''}^{<\alpha^\Downarrow} = 1 \wedge \alpha S_{\tau'_2}^{<\alpha^\Downarrow} = S); \end{aligned}$$

otherwise S on α receives *unit characteristic*.

IV. And in the last turn this definition forms the *closing condition* for S on α :

If S is *zero* matrix on α and its admissible disseminator δ falls in some maximal block of type η' below α^\Downarrow

$$[\gamma_{\tau'_1}^{<\alpha^\Downarrow}, \gamma_{\tau'_3}^{<\alpha^\Downarrow}[$$

severe for S on α , that is if there holds

$$\gamma_{\tau'_1}^{<\alpha^\Downarrow} \leq \delta < \gamma_{\tau'_3}^{<\alpha^\Downarrow} \wedge A_4^{Mb\triangleleft\alpha^\Downarrow}(\tau'_1, \tau''_1, \tau'_2, \tau'_3, \eta', \alpha S_f^{<\alpha^\Downarrow}, a_f^{<\alpha^\Downarrow})$$

below α^\Downarrow , then there is demanded the admissible data base ρ of the disseminator δ of S on α but only such that

$$\eta' < \rho \vee \rho = \chi^{*+}.$$

So, this case *hampers* the using of such S on α considerably; besides that S on α must be nonsuppressed; in any other cases no requirements are inflicted on S on α .

But remind, that the base $\rho = \chi^{*+}$ and every *unit* matrix are always admissible and nonsuppressed; every matrix is nonsuppressed for $\gamma \notin SIN_n$ in any case.

After that this definition forms the conjunction $\alpha\mathbf{K}^{<\alpha_1}$:

$$\mathbf{K}_n^{\forall < \alpha_1}(\gamma, \alpha^\Downarrow) \wedge \alpha\mathbf{K}_{n+1}^{\exists \triangleleft \alpha_1}(a, \delta, \gamma, \alpha, \rho, S) \wedge \alpha < \alpha_1$$

where is required in addition, as usual, that α^\Downarrow preserves all $SIN_n^{<\alpha_1}$ -cardinals $\leq \gamma$ below α_1 ; and, at last, there arises the formula $\alpha\mathbf{K}^{* < \alpha_1}$ from this formula $\alpha\mathbf{K}^{<\alpha_1}$ under the requirement of the nonsuppression of zero matrix S on α below α_1 .

Since definition 8.3 of the matrix α -function and of the accompanying functions follows definition 8.2, there holds the next obvious lemma, which actually repeats this definition. Here is used the notion of generating eigendisseminator $\check{\delta}^S$ for arbitrary matrix S on a carrier α , that is, remind, the minimal disseminator for S on α with the minimal possible base $\rho^S = \hat{\rho}_1$, $\rho_1 = Od(S)$ (see [27], [17]).

Lemma 8.5

Let S be an arbitrary α -matrix reduced to χ^* of characteristic a on a carrier $\alpha < \alpha_1$, admissible for $\gamma_\tau^{<\alpha_1}$ along with its disseminator $\check{\delta}$, generating disseminator $\check{\delta}$ with a base ρ , and generating eigendisseminator $\check{\delta}^S$ below α_1 , then for the prejump cardinal α^\Downarrow after χ^* there holds below α_1 :

- 1) $\forall \gamma \leq \gamma_\tau^{<\alpha_1} (SIN_n^{<\alpha_1}(\gamma) \longrightarrow SIN_n^{<\alpha^\Downarrow}(\gamma))$;
- 2) $\chi^* < \check{\delta} < \gamma_\tau^{<\alpha_1} < \alpha^\Downarrow \wedge S \triangleleft \rho \leq \chi^{*+} \wedge \rho = \hat{\rho}$;
- 3) $\check{\delta} \in SIN_n^{<\alpha^\Downarrow} \cap SIN_{n+1}^{<\alpha^\Downarrow} [< \rho]$; analogously for $\check{\delta}$;
- 4) $\sup SIN_n^{<\alpha^\Downarrow} = \alpha^\Downarrow \wedge cf(\alpha^\Downarrow) \geq \chi^{*+}$;

- 5) $a = 0 \longleftrightarrow \exists \tau'_1, \tau'_2, \tau'_3 \left(A_2^{0 \triangleleft \alpha^\downarrow}(\tau'_1, \tau'_2, \tau'_3, \alpha S_f^{< \alpha^\downarrow}) \wedge \right.$
 $\left. \wedge \forall \tau'' \in]\tau'_1, \tau'_2] a_{\tau''}^{< \alpha^\downarrow} = 1 \wedge \alpha S_{\tau'_2}^{< \alpha^\downarrow} = S \right);$
- 6) $a = 0 \longrightarrow \forall \tau'_1, \tau''_1, \tau'_2, \tau'_3, \eta' [\gamma_{\tau'_1}^{< \alpha^\downarrow} \leq \tilde{\delta} < \gamma_{\tau'_3}^{< \alpha^\downarrow} \wedge$
 $\wedge A_4^{Mb \triangleleft \alpha^\downarrow}(\tau'_1, \tau''_1, \tau'_2, \tau'_3, \eta', \alpha S_f^{< \alpha^\downarrow}, a_f^{< \alpha^\downarrow}) \longrightarrow \eta' < \rho \vee \rho = \chi^{*+}];$
- 7) (i) $\check{\delta}^S \leq \check{\delta} \leq \tilde{\delta} < \gamma_\tau^{< \alpha_1};$

(ii) if $\tilde{\delta}$ is the minimal floating disseminator of S on α with the minimal base ρ admissible for $\gamma_\tau^{< \alpha_1}$ along with ρ , then:

$$a = 1 \longrightarrow \tilde{\delta} = \check{\delta}^S \wedge \rho = \rho^S = \hat{\rho}_1, \wedge \rho_1 = Od(S),$$

that is when S is the unit matrix on α , then $\tilde{\delta}$ is the generating eigendissemator $\check{\delta}^S$ of S on α with the base ρ^S ;

- 8) there exist the minimal carrier $\alpha' < \gamma_{\tau+1}^{< \alpha_1}$ of S of the same characteristic a admissible for $\gamma_\tau^{< \alpha_1}$ along with the same accessories $\tilde{\delta}, \rho$ below α_1 :

$$\gamma_\tau^{< \alpha_1} < \alpha' < \gamma_{\tau+1}^{< \alpha_1};$$

analogously for nonsuppressibility of S for γ_τ along with its accessories.

Proof. It remains to prove the last two statements; the upper index $< \alpha_1$ will be dropped.

So, let us consider the matrix S of characteristic a on its carrier

$\alpha < \alpha_1$ admissible for $\gamma_\tau^{<\alpha_1}$ along with its disseminator $\tilde{\delta}$ and base ρ . Statement 7) (i) is obvious; as to 7) (ii) let us consider $a = 1$, then the base

$$\rho = \rho^S = \hat{\rho}_1, \rho_1 = Od(S)$$

along with the minimal disseminator

$$\check{\delta}^S \in SIN_n^{<\alpha^\Downarrow} \cap SIN_{n+1}^{<\alpha^\Downarrow} [< \rho^S]$$

evidently fulfill all requirements of the condition

$$\alpha \mathbf{K}(a, \check{\delta}^S, \gamma, \alpha, \rho^S, S)$$

up to the last it conjunctive constituent \mathbf{K}^0 .

But the latter is fulfilled also, because for $a = 1$ its premise fails. Thus the whole $\alpha \mathbf{K}$ is fulfilled and $\tilde{\delta} = \check{\delta}^S, \rho = \rho^S$.

Turning to the proof of 8) it is not hard to apply lemma 3.2 [27] (about restriction) just as it was done in the lemma 5.17 2) (ii) proof. Nevertheless, this application presents the typical reasoning, which will be used further in various important cases, so one should accept it in details.

First, it was assumed above that α_1 is limit for the class $SIN_{n-1}^{<\alpha_1}$ (remind the convention after (7.1)), therefore it always exist $\gamma_{\tau+1}^{<\alpha_1}$ for every $\gamma_\tau^{<\alpha_1}$.

Next, suppose that the matrix S with the disseminator δ and base ρ on the carrier

$$\alpha \in]\gamma_\tau^{<\alpha_1}, \alpha_1[$$

is admissible for $\gamma_\tau^{<\alpha_1}$ below α_1 , then it holds the following proposition $\varphi(\chi^*, \delta, \gamma_\tau^{<\alpha_1}, \rho, S)$:

$$\exists \alpha' (\gamma_\tau^{<\alpha_1} < \alpha' \wedge \alpha \mathbf{K}(\delta, \gamma_\tau^{<\alpha_1}, \alpha', \rho, S))$$

below α_1 , that is after its \triangleleft -bounding by the cardinal α_1 . This proposition φ itself is from the class Σ_n , because it includes Σ_n -formula \mathbf{K}_n^\forall . But let us use the cardinal

$$\gamma_{\tau^n} = \sup \{ \gamma \leq \gamma_\tau^{<\alpha_1} : \gamma \in SIN_n^{<\alpha_1} \};$$

by lemma 3.4 [27] γ_{τ^n} belongs to $SIN_n^{<\alpha_1}$ too. Now let us replace in the formula $\alpha\mathbf{K}$ its subformula \mathbf{K}_n^\forall with the Δ_1 -formula

$$SIN_n^{<\alpha^\downarrow}(\gamma_{\tau^n}),$$

then the Σ_n -formula $\alpha\mathbf{K}$ turns into some Π_{n-2} -formula, which we shall denote through $\alpha\mathbf{K}_{n-2}$. Consequently, the formula φ turns into some Σ_{n-1} -formula $\varphi_{n-2}(\chi^*, \delta, \gamma_{\tau^n}, \gamma_\tau^{<\alpha_1}, \rho, S)$:

$$\exists \alpha' (\gamma_\tau^{<\alpha_1} < \alpha' \wedge \alpha\mathbf{K}_{n-2}(\delta, \gamma_{\tau^n}, \gamma_\tau^{<\alpha_1}, \alpha', \rho, S))$$

precisely with the same content below α_1 , and there holds

$$\varphi_{n-2}^{<\alpha_1}(\chi^*, \delta, \gamma_{\tau^n}, \gamma_\tau^{<\alpha_1}, \rho, S).$$

The last proposition contains individual constants

$$\chi^*, \delta, \gamma_{\tau^n}, \gamma_\tau^{<\alpha_1}, \rho, S$$

less than the $SIN_{n-1}^{<\alpha_1}$ -cardinal $\gamma_{\tau+1}^{<\alpha_1}$ and therefore this cardinal restricts this proposition by lemma 3.2 [27] (where n replaced with $n-1$), that is there holds the formula

$$\exists \alpha' \in [\gamma_\tau^{<\alpha_1}, \gamma_{\tau+1}^{<\alpha_1}[\quad \alpha\mathbf{K}_{n-2}^{<\alpha_1}(\delta, \gamma_{\tau^n}, \gamma_\tau^{<\alpha_1}, \alpha', \rho, S)$$

and S receives its carrier $\alpha' \in [\gamma_\tau^{<\alpha_1}, \gamma_{\tau+1}^{<\alpha_1}[$ admissible for $\gamma_\tau^{<\alpha_1}$ along with its previous disseminator and data base.

The part 8) for nonsuppressibility will not be used up to §11 and there we shall return to it once more.

—

It is not hard to see that the functions introduced above in definition 8.3 possess many simple properties of δ -functions and their accompanying functions, so the proofs of the following three lemmas are analogous to those of lemmas 7.3, 7.4 (or lemmas 5.16, 5.15 [27]) and lemma 7.5:

Lemma 8.6

For $\alpha_1 < k$ the formulas $\alpha\mathbf{K}^{<\alpha_1}$, $\alpha\mathbf{K}^{*<\alpha_1}$ belong to Δ_1 and therefore all functions from definition 8.3:

$$a_f^{<\alpha_1}, \alpha S_f^{<\alpha_1}, \tilde{\delta}_f^{<\alpha_1}, \rho_f^{<\alpha_1}, \alpha_f^{<\alpha_1}, \check{\delta}_f^{<\alpha_1}$$

are Δ_1 -definable through χ^*, α_1 . For $\alpha_1 = k$ the formula $\alpha\mathbf{K}$ belongs to Σ_{n+1} .

—

Lemma 8.7 (About α -function absoluteness)

Let $\chi^* < \gamma_{\tau+1}^{<\alpha_1} < \alpha_2 < \alpha_1 \leq k$, $\alpha_2 \in SIN_{n-2}^{<\alpha_1}$ and

$$(\gamma_\tau^{<\alpha_1} + 1) \cap SIN_n^{<\alpha_2} = (\gamma_\tau^{<\alpha_1} + 1) \cap SIN_n^{<\alpha_1}.$$

1) Then on the set

$$T = \{\tau' : \chi^* \leq \gamma_{\tau'}^{<\alpha_2} \leq \gamma_\tau^{<\alpha_1}\}$$

the admissibility below α_2 coincides with the admissibility below α_1 : for every $\tau' \in T$ and a matrix S' on its carrier $\alpha' \in]\gamma_{\tau'}^{<\alpha_2}, \gamma_{\tau+1}^{<\alpha_1}[$

$$\alpha\mathbf{K}^{<\alpha_2}(\gamma_{\tau'}^{<\alpha_2}, \alpha', S') \leftrightarrow \alpha\mathbf{K}^{<\alpha_1}(\gamma_{\tau'}^{<\alpha_2}, \alpha', S');$$

2) on the set

$$\{\tau' : \chi^* \leq \gamma_{\tau'}^{<\alpha_2} \leq \gamma_\tau^{<\alpha_1} \wedge (a_{\tau'}^{<\alpha_2} = 1 \vee \neg SIN_n^{<\alpha_2}(\gamma_{\tau'}^{<\alpha_2}))\}$$

the functions (8.2) below $\alpha^0 = \alpha_2$ coincide respectively with the functions (8.2) below $\alpha^0 = \alpha_1$.

—

Lemma 8.8 (About disseminator)

1) *Let*

$$(i) \quad]\tau_1, \tau_2[\subseteq \text{dom}(\alpha S_f^{<\alpha_1}), \quad \gamma_{\tau_2} \in SIN_n^{<\alpha_1};$$

$$(ii) \quad \tau_3 \in \text{dom}(\alpha S_f^{<\alpha_1}), \quad \tau_2 \leq \tau_3;$$

$$(iii) \quad \tilde{\delta}_{\tau_3}^{<\alpha_1} < \gamma_{\tau_2}^{<\alpha_1} \quad \text{and} \quad a_{\tau_3}^{<\alpha_1} = 0.$$

Then

$$\tilde{\delta}_{\tau_3}^{<\alpha_1} \leq \gamma_{\tau_1}^{<\alpha_1}.$$

Analogously for $\check{\delta}_{\tau_3}^{<\alpha_1}$.

2) *Let* α -*matrix* S *of characteristic* a *on a carrier* α *be admissible for* $\gamma_{\tau}^{<\alpha_1}$ *along with its disseminator* $\tilde{\delta}$ *and base* ρ *below* α_1 , *then*

$$\{\tau' : \tilde{\delta} < \gamma_{\tau'}^{<\alpha_1} < \gamma_{\tau}^{<\alpha_1}\} \subseteq \text{dom}(\alpha S_f^{<\alpha_1}).$$

Proof. 1) The reasoning forthcoming is analogous to the proof of lemma 7.5 1), but now the special properties of matrix disseminator of unit or zero characteristic involves the singular situation. Therefore, here one should use the following argument that will be applied further in various significant typical situations; here it is presented in outline; the upper indices $< \alpha_1$, $\triangleleft \alpha_1$ will be dropped for shortness.

Suppose 1) fails; let us consider the matrix $S^3 = \alpha S_{\tau_3}$ of characteristic $a^3 = a_{\tau_3} = 0$ on the carrier α_{τ_3} with the prejump cardinal $\alpha^3 = \alpha_{\tau_3}^{\downarrow}$, possessing the disseminators $\check{\delta}^3 = \check{\delta}_{\tau_3}$, $\tilde{\delta}^3 = \tilde{\delta}_{\tau_3}$ with the base $\rho^3 = \rho_{\tau_3}$, and suppose that

$$\gamma_{\tau_1} < \tilde{\delta}^3 < \gamma_{\tau_2}. \quad (8.3)$$

One should consider here the minimal ordinal τ_1 with the property (i).

By definition 8.3 the proposition $\alpha\mathbf{K}$ holds and hence there holds the proposition \mathbf{K}^0 :

$$a^3 = 0 \longrightarrow \forall \tau'_1, \tau''_1, \tau'_2, \tau'_3, \eta' < \alpha^3 \left[\gamma_{\tau'_1}^{<\alpha^3} \leq \tilde{\delta}^3 < \gamma_{\tau'_3}^{<\alpha^3} \wedge \right. \\ \left. \wedge A_4^{Mb < \alpha^3}(\tau'_1, \tau''_1, \tau'_2, \tau'_3, \eta', \alpha S_f^{<\alpha^3}, a_f^{<\alpha^3}) \longrightarrow \eta' < \rho^3 \vee \rho^3 = \chi^{*+} \right].$$

Suppose that there exist some ordinals $\tau'_1, \tau''_1, \tau'_2, \tau'_3, \eta' < \alpha^3$ that fulfill the premise of this proposition:

$$\gamma_{\tau'_1}^{<\alpha^3} \leq \tilde{\delta}^3 < \gamma_{\tau'_3}^{<\alpha^3} \wedge A_4^{Mb < \alpha^3}(\tau'_1, \tau''_1, \tau'_2, \tau'_3, \eta', \alpha S_f^{<\alpha^3}, a_f^{<\alpha^3}) . \quad (8.4)$$

It should be pointed out again that due to $A_4^{Mb < \alpha^3}$ these ordinals are defined through $\tilde{\delta}^3, \alpha^3$ *uniquely*. Since $\gamma_{\tau_2} \in SIN_n$ and γ_{τ_1} is the minimal one can see, that due to supposition (8.3) there comes

$$\gamma_{\tau'_1} < \tilde{\delta}^3 = \tilde{\delta}^3 < \gamma_{\tau_2} \quad (8.5)$$

as a result of the minimizing of the disseminator $\tilde{\delta}_{\tau_3}$ inside the interval $[\gamma_{\tau'_1}, \gamma_{\tau_2}]$ according to definition 8.3. Now we come to the situation from the proof of lemma 7.5 1) and it remains to repeat its arguments and to use the \leq -minimal matrix $S^m < S^3$ on some carrier $\alpha^m \in]\gamma_{\tau_3}, \alpha^3[$ of characteristic a^m , admissible and nonsuppressed for γ_{τ_3} along with its minimal disseminator $\tilde{\delta}^m < \gamma_{\tau_2}$ and base $\rho^m < Od(S^3)$, because the suppression of S^m for γ_{τ_3} implies the suppression of the matrix S^3 itself for γ_{τ_3} , though it is nonsuppressed by definition (below α_1).

Now there comes the contradiction: since $S^m < S^3$ and $a^3 = 0$ then by definition 8.3 the matrix S^3 cannot be the minimal value αS_{τ_3} .

If there is no such ordinals $\tau'_1, \tau''_1, \tau'_2, \tau'_3, \eta'$, the proposition \mathbf{K}^0 survives evidently under minimizing the disseminator $\tilde{\delta}_{\tau_3}$

inside $\underset{\sim}{\gamma}_{\tau_1}, \underset{\sim}{\gamma}_{\tau_2}[$ and so $\underset{\sim}{\delta}_{\tau_3} \leq \gamma_{\tau_1}$, otherwise again there holds $\gamma_{\tau_1} < \underset{\sim}{\delta}_{\tau_3} = \underset{\sim}{\delta}_{\tau_3} < \gamma_{\tau_2}$ and the same reasoning provides the same contradiction.

Turning to 2), one should simply notice, that this statement repeats the previous lemmas 5.17 2) [27], 7.5 2) in the following form:

the matrix S being admissible for γ_τ on its carrier α_τ , by lemma 8.5 8) and definition 8.2 remains still admissible and non-suppressed for every $\gamma_{\tau'} < \gamma_\tau$, such that $\underset{\sim}{\delta} < \gamma_{\tau'}$, along with the same accompanying ordinals $a, \underset{\sim}{\delta}, \rho, \alpha$. For the unit characteristic $a = 1$ it is obvious; but for $a = 0$ this lemma will be used only in §11 and there we shall turn to its proof in the exposition detailed more. \dashv

The following lemmas confirm the further extension of the δ -functions theory on α -functions and are analogous to lemmas 7.6, 7.7 about δ -function definiteness on the final subinterval of the inaccessible cardinal k .

So, the next lemma shows, that there exists the cardinal $\delta < k$ such that

$$\{\tau : \delta < \gamma_\tau < k\} \subseteq \text{dom}(\alpha S_f);$$

more precisely:

Lemma 8.9 (About α -function definiteness)

There exist cardinals $\delta < \gamma < k$ such that for every SIN_n -cardinal $\alpha_1 > \gamma$, $\alpha_1 < k$ limit for $SIN_n \cap \alpha_1$ the function $\alpha S_f^{<\alpha_1}$ is defined on the nonempty set

$$T^{\alpha_1} = \{\tau : \delta < \gamma_\tau^{<\alpha_1} < \alpha_1\}.$$

The minimal of such cardinals δ is denoted by $\alpha\delta^$, its successor in SIN_n by $\alpha\delta^{*1}$ and the following corresponding ordinals are*

introduced:

$$\alpha\tau_1^* = \tau(\alpha\delta^*), \quad \alpha\tau^{*1} = \tau(\alpha\delta^{*1}),$$

$$\text{so that } \alpha\delta^* = \gamma_{\alpha\tau_1^*}, \quad \alpha\delta^{*1} = \gamma_{\alpha\tau^{*1}},$$

$$\text{and } \alpha^{*1} = \alpha_{\alpha\tau^{*1}}^{<\alpha_1\Downarrow}, \quad \alpha\rho^{*1} = \rho_{\alpha\tau^{*1}}^{<\alpha_1}.$$

Proof consists in the application of lemma 6.14 [27] just as it was done in the proof of lemma 7.6 but for the *greater* reducing cardinal

$$\chi = (\chi^*)^{+\omega_0} \quad \text{and} \quad \alpha_1 = k, \quad m = n + 1.$$

The resulting function \mathfrak{A} , being defined on the set

$$T = \{\tau : \gamma_{\tau_0} \leq \gamma_\tau < k\},$$

should be treated in the following way:

Let us consider by lemma 6.14 [27] the matrix $S_\tau^1 = \mathfrak{A}(\tau)$ reduced to the cardinal $\chi = (\chi^*)^{+\omega_0}$ on the carrier $\alpha_\tau^1 > \gamma_\tau$; it has the admissible generating eigendisseminator $\check{\delta}_\tau^1 < \gamma_\tau$ with the base $\rho_\tau^1 \triangleright S_\tau^1$. One can see that $\rho_\tau^1 > \chi^{*+}$ and so $\check{\delta}_\tau^1$ can be considered as the disseminator for S_τ^1 on α_τ^1 with the base χ^{*+} . Now let us turn to the prejump cardinal

$$\alpha^1 = \alpha_\tau^{1\Downarrow};$$

by the same lemma $cf(\alpha^1) \geq \chi^{*+}$ and it is possible to introduce the matrix S_τ^2 reduced to χ^* possessing the same prejump cardinal α^1 and hence the same disseminator δ_τ^1 with the same base χ^{*+} using lemma 5.12 [27] in the following way:

If there holds the proposition

$$\exists \alpha \in [\alpha^1, \alpha_\tau^1[\quad \sigma(\chi^*, \alpha) \tag{8.6}$$

then let S_τ^2 be the matrix reduced to χ^* on the minimal carrier α_τ^2 and produced by the cardinal α^1 , so that $\alpha^1 = \alpha_\tau^{2\Downarrow}$ (just as it was done in the proof of lemma 6.12 [27] by the cardinal α_0 , playing the role of α^1 here).

In the opposite case, when (8.6) fails, one can see that, since the proposition of lemma 5.12 [27] is fulfilled below α^1 , the matrix S_τ^1 protects the jump cardinal α_τ^1 (and, hence, α^1) which is preserves under the reducing the matrix S_τ^1 on the carrier α_τ^1 to χ^* ; so we can define the matrix (see definitions 4.1, 5.1, 5.5 [27])

$$S_\tau^2 \Rightarrow \widetilde{\mathbf{S}}_n^{\sin \triangleleft \alpha_\tau^1} \bar{\bar{\chi}}^* \quad \text{on the carrier} \quad \alpha_\tau^2 = \sup \text{dom}(\widetilde{\mathbf{S}}_n^{\sin \triangleleft \alpha_\tau^1} \bar{\bar{\chi}}^*).$$

This matrix S_τ^2 is singular on the carrier α_τ^2 : conditions 1), 3) of definition 5.7 [27] are obvious, while condition 2) one can verify with the help of the *splitting method*, repeating the argument from the proof of lemma 5.12 [27] (where α_1 , χ are replaced with α_τ^2 , χ^* respectively) literally.

In any case $\alpha^1 = \alpha_\tau^{2\Downarrow}$ and S_τ^2 is found to be admissible on α_τ^2 for γ_τ along with the same disseminator $\check{\delta}_\tau^1$ and its base χ^{*+} , because all conditions of \mathbf{K}^0 from definition 8.2 trivially holds when $\rho = \chi^{*+}$. Also such matrix S_τ^2 on its carrier α_τ^2 is non-suppressed due to its base $\rho = \chi^{*+}$. It can be unit or zero, but in any case there exist some α -matrix reduced to χ^* admissible and nonsuppressed for γ_τ along with its accessories.

Now one should take any cardinal γ great enough and such that for any $\gamma_\tau > \gamma$ there exist some matrix S_τ^2 with the base $\rho = \chi^{*+}$; it is admissible and nonsuppressed for γ_τ below α_1 for any $\alpha_1 \in \text{SIN}_n$ by definition.

So, after the minimizing such resulting matrices and their accompanying ordinals according to definition 8.3 there appears the function $\alpha S_f^{< \alpha_1}$ and accompanying ordinal functions defined on T^{α_1} for any $\alpha_1 \in \text{SIN}_n$, $\alpha_1 > \gamma$.

—

In conclusion of this section repeating the proof mode of lemma 7.7 it is easy to draw out

Lemma 8.10

$$\alpha\delta^* \in SIN_n \cap SIN_{n+1}^{<\alpha^*1} [<\alpha\rho^*1].$$

Proof. Let us use the designations from the previous lemma 8.9. First starts lemma 7.7 proof mode, treating the disseminator $\tilde{\delta}_{\alpha\tau^*1}$ with the base $\alpha\rho^*1$ of the matrix $\alpha S_{\alpha\tau^*1}$ on the carrier $\alpha_{\alpha\tau^*1}$ with the prejump cardinal $\alpha^*1 = \alpha_{\alpha\tau^*1}^\downarrow$. Since

$$\alpha\delta^*1 \in SIN_n, \tilde{\delta}_{\alpha\tau^*1} < \alpha\delta^*1$$

and

$$\tilde{\delta}_{\alpha\tau^*1} \in SIN_n^{<\alpha^*1} \cap SIN_{n+1}^{<\alpha^*1} [<\alpha\rho^*1],$$

lemma 3.8 implies $\tilde{\delta}_{\alpha\tau^*1} \in SIN_n$.

Now suppose, that this lemma 8.10 is wrong and

$$\alpha\delta^* \notin SIN_n,$$

then

$$\tilde{\delta}_{\alpha\tau^*1} < \alpha\delta^* = \gamma_{\alpha\tau_1^*}.$$

Due to lemma 3.2 [27] it makes possible to restrict the Σ_{n-1} -proposition about existence of matrix $\alpha S_{\alpha\tau^*1}$ admissible carrier along with the same $\tilde{\delta}_{\alpha\tau^*1}$, $\alpha\rho^*1$ to the SIN_{n-1} -cardinal $\gamma_{\alpha\tau_1^*+1}$, just as it was done in the proof of lemma 8.5 8).

Then the matrix $\alpha S_{\alpha\tau^*1}$ receives again some its carrier

$$\alpha' \in]\gamma_{\alpha\tau^*1}, \gamma_{\alpha\tau^*1+1}[$$

admissible for $\gamma_{\alpha\tau^*1}$ along with its previous disseminator and data base.

But due to the minimality of $\alpha\tau_1^*$ there holds

$$\alpha\tau_1^* \notin \text{dom}(\alpha S_f).$$

It can be only when αS_{τ^*1} on α' is admissible but suppressed for $\gamma_{\alpha\tau_1^*}$; in its turn it can be only when

$$\alpha\delta^* = \gamma_{\alpha\tau_1^*} \in SIN_n$$

contrary to the supposition.

As to the rest part of this lemma:

$$\alpha\delta^* \in SIN_n^{<\alpha^*1} [<\alpha\rho^{*1}],$$

it is not needed in what follows up to §11 and therefore we shall return to it there. \dashv

9 Analysis of α -Function Monotonicity

Here the first component of the required contradiction – the monotonicity of α -function – is investigated in various important cases.

As we shall see, this property is rather strong; in particular, any interval $[\tau_1, \tau_2[$ of its monotonicity can not be “too long”, – the corresponding interval $]\gamma_{\tau_1}, \gamma_{\tau_2}[$ can not contain any SIN_n -cardinals, and if $\gamma_{\tau_2} \in SIN_n$, then this function receive some constant characteristics and *stabilizes* on such $[\tau_1, \tau_2[$.

We start with the latter situation:

Definition 9.1

The function $\alpha S_f^{<\alpha_1}$ is called monotone on interval $[\tau_1, \tau_2[$ and on the corresponding interval $[\gamma_{\tau_1}^{<\alpha_1}, \gamma_{\tau_2}^{<\alpha_1}[$ below α_1 iff $\tau_1 + 1 < \tau_2$, $]\tau_1, \tau_2[\subseteq \text{dom}(\alpha S_f^{<\alpha_1})$ and

$$\forall \tau', \tau'' (\tau_1 < \tau' < \tau'' < \tau_2 \longrightarrow \alpha S_{\tau'}^{<\alpha_1} \leq \alpha S_{\tau''}^{<\alpha_1}) .$$

⊢

To operate with this notion it is suitable to use the following Δ_1 -formulas, which play the main role in this section:

$$A_0^{1 \leq \alpha_1}(\chi, \tau_1, \tau_2, \alpha S_f^{<\alpha_1}):$$

$$A_0^{<\alpha_1}(\chi, \tau_1, \tau_2, \alpha S_f^{<\alpha_1}) \wedge \forall \tau', \tau'' (\tau_1 < \tau' < \tau'' < \tau_2 \rightarrow \alpha S_{\tau'}^{<\alpha_1} \leq \alpha S_{\tau''}^{<\alpha_1});$$

so, here is stated, that $\alpha S_f^{<\alpha_1}$ is defined on the interval $]\tau_1, \tau_2[$ with the property A_0 (remind definition 8.1 1.0) and, moreover, it is monotone on the interval $[\tau_1, \tau_2[$; thus we shall name it and the corresponding interval $[\gamma_{\tau_1}, \gamma_{\tau_2}[$ the intervals of the function

$\alpha S_f^{<\alpha_1}$ monotonicity;

$$A_1^{1<\alpha_1}(\chi, \tau_1, \tau_2, \alpha S_f^{<\alpha_1}):$$

$$A_0^{1<\alpha_1}(\chi, \tau_1, \tau_2, \alpha S_f^{<\alpha_1}) \wedge \exists \gamma^2 (\gamma^2 = \gamma_{\tau_2} \wedge SIN_n^{<\alpha_1}(\gamma^2));$$

further the function $\alpha S_f^{<\alpha_1}$ will be omitted in such notations for brevity (if it will be pointed out in the context).

Now it is not still all ready to prove the total monotonicity of matrix function αS_f , but some its fragments are clear quite analogous to lemmas 5.17 1) [27], 7.9. For instance, from lemma 3.2 [27] it comes directly

Lemma 9.2 (About α -function monotonicity)

Let

$$\tau_1 < \tau_2, \quad a_{\tau_2}^{<\alpha_1} = 1 \quad \text{and} \quad \tilde{\delta}_{\tau_2}^{<\alpha_1} < \gamma_{\tau_1},$$

then

$$\alpha S_{\tau_1}^{<\alpha_1} \leq \alpha S_{\tau_2}^{<\alpha_1} \quad \text{and} \quad a_{\tau_1}^{<\alpha_1} = 1.$$

Analogously for zero characteristic $a_{\tau_1}^{<\alpha_1} = a_{\tau_2}^{<\alpha_1} = 0$.

—

Lemma 9.3 (About α -function stabilization)

Let

(i) $\alpha S_f^{<\alpha_1}$ be monotone on $[\tau_1, \tau_2[$ below α_1 :

$$A_1^{1<\alpha_1}(\tau_1, \tau_2);$$

(ii) $\gamma_{\tau_2}^{<\alpha_1}$ be a successor in $SIN_n^{<\alpha_1}$.

Then $\alpha S_f^{<\alpha_1}$ stabilizes on $[\tau_1, \tau_2[$, that is there exist S^0 and $\tau_0 \in]\tau_1, \tau_2[$ such that

$$\forall \tau \in [\tau_0, \tau_2[\quad \alpha S_{\tau}^{<\alpha_1} = S^0.$$

The least of such ordinals τ_0 is called the stabilization ordinal of $\alpha S_f^{<\alpha_1}$ for τ_2 below α_1 and denoted through $\tau_2^{s<\alpha_1}$.

Proof represents once more the typical application of lemma 3.2 [27]; we shall omit the upper indices $<\alpha_1, \triangleleft \alpha_1$. Suppose this lemma is wrong; let us consider the ordinal

$$\rho^0 = \sup\{Od(\alpha S_\tau) : \tau_1 < \tau < \tau_2\}.$$

Let us apply the mode of reasoning used above in the proof of lemma 8.5 8) and introduce the cardinals

$$\gamma_{\tau_2^n} = \sup\{\gamma < \gamma_{\tau_2} : \gamma \in SIN_n\};$$

$$\gamma_{\tau_{1,2}^n} = \max\{\gamma_{\tau_1}, \gamma_{\tau_2^n}\}.$$

Then one should repeat definition 8.3 of the matrix function αS_f and its accompanying ordinal functions below α_1 on the set

$$T_{\tau_{1,2}^n}^{\alpha_1} = \{\tau : \gamma_{\tau_{1,2}^n} < \gamma_\tau < \alpha_1\} \quad (9.1)$$

but preserving *only* SIN_n -cardinals $\leq \gamma_{\tau_2^n}$; it can be done in the following way:

Definition 8.3 is based on the formula

$$\alpha \mathbf{K}^*(a, \delta, \gamma_\tau, \alpha, \rho, S) \quad (9.2)$$

below α_1 (see definition 8.2 4)), which means, that S is the α -matrix on its carrier α of characteristic a with the disseminator δ and base ρ admissible for γ_τ and, moreover, nonsuppressed on this α for γ_τ below α_1 ; but since for every $\tau \in T_{\tau_{1,2}^n}^{<\alpha_1}$ there holds $\gamma_\tau \notin SIN_n$, the nonsuppression condition $\neg A_5^{S,0}$ in $\alpha \mathbf{K}^*$ holds on and it can be dropped, and then $\alpha \mathbf{K}^*$ turns into the formula $\alpha \mathbf{K}$.

This formula is from the class Σ_n , because it includes the Σ_n -formula \mathbf{K}_n^\forall . But let us use the cardinal $\gamma_{\tau_2^n}$ and replace in formula (9.2) its subformula \mathbf{K}_n^\forall with Δ_1 -formula

$$SIN_n^{<\alpha^\Downarrow}(\gamma_{\tau_2^n}),$$

then the Σ_n -formula (9.2) turns into some Π_{n-2} -formula, which we shall denote through

$$\alpha\mathbf{K}_{n-2}^{*1}(a, \delta, \gamma_\tau, \alpha, \rho, S).$$

So, the matrix function defined on the set $T_{\tau_{1,2}^n}^{\alpha_1}$ (9.1) as in definition 8.3, but through the formula (9.2) replaced with $\alpha\mathbf{K}_{n-2}^{*1}$, evidently coincides with the function αS_f on the interval $]\tau_{1,2}^n, \tau_2[$; we shall denote it by αS_f^1 .

Now, since αS_f^1 is monotone on $]\tau_{1,2}^n, \tau_2[$ but is not stabilized on this interval, the ordinal ρ_0 is *limit* and there holds the following proposition below γ_{τ_2} :

$$\forall \tau \ (\tau_{1,2}^n < \tau \longrightarrow \exists S \ (S = \alpha S_\tau^1 \wedge S \triangleleft \rho^0)) .$$

It can be formulated in the Π_n -form:

$$\begin{aligned} & \forall \tau, \gamma', \gamma'' \left[\gamma_{\tau_{1,2}^n} < \gamma' = \gamma_\tau < \gamma'' = \gamma_{\tau+1} \rightarrow \right. \\ & \rightarrow \left(\exists \delta, \alpha, \rho < \gamma'' \exists S \triangleleft \rho \ (\alpha\mathbf{K}_{n-2}^{*1}(1, \delta, \gamma', \alpha, \rho, S) \wedge S \triangleleft \rho^0) \vee \right. \\ & \vee \left(\exists \delta, \alpha, \rho < \gamma'' \exists S \triangleleft \rho \ \alpha\mathbf{K}_{n-2}^{*1}(0, \delta, \gamma', \alpha, \rho, S) \wedge S \triangleleft \rho^0 \wedge \right. \\ & \left. \left. \wedge \forall \delta', \alpha', \rho' < \gamma'' \forall S' \triangleleft \rho' \neg \alpha\mathbf{K}_{n-2}^{*1}(1, \delta', \gamma', \alpha', \rho', S') \right) \right) \left. \right] . \end{aligned} \quad (9.3)$$

Now it comes the contradiction:

On one hand, the SIN_n -cardinal γ_{τ_2} extends this proposition (9.3) up to α_1 and as a result the matrix $\alpha S_{\tau_2}^1 \triangleleft \rho^0$ arises.

But, on the other hand, ρ^0 is the limit ordinal and there exists

$$\tau_{1,3}^n \in]\tau_{1,2}^n, \tau_2[\text{ such that } \alpha S_{\tau_{1,3}^n}^1 \succ \alpha S_{\tau_2}^1.$$

That is why below γ_{τ_2} the next proposition holds:

$$\forall \tau \left(\tau_{1,3}^n < \tau \longrightarrow \forall S (S = \alpha S_{\tau}^1 \longrightarrow S \succ \alpha S_{\tau_2}^1) \right).$$

It also can be formulated in Π_n -form:

$$\begin{aligned} & \forall \tau, \gamma', \gamma'' \left[\gamma_{\tau_{1,3}^n}^n < \gamma' = \gamma_{\tau} < \gamma'' = \gamma_{\tau+1} \rightarrow \right. \\ & \rightarrow \left(\forall \delta, \alpha, \rho < \gamma'' \forall S \triangleleft \rho (\alpha \mathbf{K}_{n-2}^{*1}(1, \delta, \gamma', \alpha, \rho, S) \rightarrow \alpha S_{\tau_2}^1 \triangleleft S) \wedge \right. \\ & \wedge \forall \delta, \alpha, \rho < \gamma'' \forall S \triangleleft \rho (\alpha \mathbf{K}_{n-2}^{*1}(0, \delta, \gamma', \alpha, \rho, S) \wedge \\ & \wedge \forall \delta', \alpha', \rho' < \gamma'' \forall S' \triangleleft \rho' \neg \alpha \mathbf{K}_{n-2}^{*1}(1, \delta', \gamma', \alpha', \rho', S') \rightarrow \\ & \left. \left. \rightarrow \alpha S_{\tau_2}^1 \triangleleft S \right) \right) \right], \end{aligned} \quad (9.4)$$

which γ_{τ_2} extends up to α_1 and therefore

$$\alpha S_{\tau_2}^1 \triangleleft \alpha S_{\tau_2}^1.$$

⊥

Let us remind that the symbols \mathbf{n}^α , χ^* , $\alpha S_f^{<\alpha_1}$, $a_f^{<\alpha_1}$ in writings

of formulas can be often omitted for shortening. Besides, the usual condition of equinformativeness $A_6^e(\alpha_1)$

$$\begin{aligned} \chi^* < \alpha_1 \wedge A_n^{<\alpha_1}(\chi^*) &= \|u_n^{<\alpha_1}(\underline{l})\| \wedge SIN_{n-2}(\alpha_1) \wedge \\ &\wedge \forall \gamma < \alpha_1 \exists \gamma_1 \in [\gamma, \alpha_1[\ SIN_n^{<\alpha_1}(\gamma_1) \end{aligned}$$

is always superimposed on the bounding cardinals α_1 .

The stabilization property is very important for what follows; moreover, it comes out that the analogous attribute arises for the characteristic function, which play the crucial role further. Complicating in a certain way the reasoning from the proof of lemma 9.3 it is possible to prove the similar characteristic property:

Lemma 9.4 (About characteristic stabilization)

Let

- (i) $A_1^{1<\alpha_1}(\tau_1, \tau_2)$;
- (ii) $\forall \tau < \tau_2 \quad \exists \tau' \in [\tau, \tau_2[\quad a_{\tau'}^{<\alpha_1} = 1$;

Then

$$\forall \tau' \in]\tau_1, \tau_2[\quad a_{\tau'}^{<\alpha_1} = 1.$$

In this case we shall say that the unit characteristic stabilizes on $[\tau_1, \tau_2[$ below α_1 .

Analogously for zero characteristic.

Proof is carried out by the induction on the pair (α_1, τ_2) ; (remind, the set of such pairs is considered to be lexicographically ordered as above, with α_1 as the first component and τ_2 as the second). Suppose this pair is minimal violating the lemma. It is not hard

to see that $\gamma_{\tau_2}^{<\alpha_1}$ is the successor in $SIN_n^{<\alpha_1}$; precisely this case is used further. Recall, that for matrices of unit characteristic on their carriers the suppression condition $A_5^{S,0}$ fails and it can be dropped for these matrices; so for the unit characteristic $a = 1$ the formula $\alpha\mathbf{K}^{*<\alpha_1}$ is equivalent to $\alpha\mathbf{K}^{<\alpha_1}$. The upper indices $<\alpha_1, \triangleleft\alpha_1$ will be dropped for some brevity.

By the previous lemma there exist the stabilization ordinal τ_2^s of αS_f on $[\tau_1, \tau_2[$, and the matrix S^0 such that

$$\forall \tau \in [\tau_2^s, \tau_2[\quad \alpha S_\tau = S^0.$$

According to the condition of this lemma there exists the minimal $\tau^1 \in [\tau_2^s, \tau_2[$ such that $a_{\tau^1} = 1$. The further reasoning splits into two parts:

1. First, let us prove that $\forall \tau \in [\tau^1, \tau_2[\quad a_\tau = 1$.

Suppose it is wrong, then there exists the minimal $\tau^0 \in]\tau^1, \tau_2[$ providing $a_{\tau^0} = 0$; thus $a_\tau \equiv 1$ on $[\tau^1, \tau^0[$ (so, take notice, the matrix S^0 on different admissible carriers can possess different characteristic here). Let us consider the following subcases for

$$S^0 = \alpha S_{\tau^0}, \quad \check{\delta}^0 = \check{\delta}_{\tau^0}, \quad \alpha^0 = \alpha_{\tau^0}^\Downarrow :$$

1a. $\check{\delta}^0 \notin SIN_n$. Since $\gamma_{\tau^1} \in SIN_n$, lemma 3.8 1) [27] implies

$$\gamma_{\tau^1} < \check{\delta}^0, \quad \check{\delta}^0 \in \left(SIN_n^{<\alpha^0} - SIN_n \right)$$

and we can use the cardinal

$$\gamma_{\tau^2} = \min \left(SIN_n^{<\alpha^0} - SIN_n \right).$$

Due to the same lemma 3.8 [27] it is not hard to see that γ_{τ^2} is the *successor* in $SIN_n^{<\alpha^0}$ of some cardinal

$$\gamma_{\tau^3} \geq \gamma_{\tau^1}, \quad \gamma_{\tau^3} \in SIN_n \text{ below } \alpha_1,$$

and the function $\alpha S_f^{<\alpha^0}$ is monotone on the interval $[\tau^3, \tau^2[$. Since $a_\tau \equiv 1$ on $[\tau^1, \tau^0[$ lemma 3.2 [27] provides that the interval $]\gamma_{\tau^1}, \gamma_{\tau^2}[$ contains admissible carriers of matrices of unit characteristic disposed cofinally to γ_{τ^2} because the $SIN_n^{<\alpha^0}$ -cardinal γ_{τ^2} restricts the Σ_n -proposition about the existence of such carriers. After that the cardinal γ_{τ^2} extends unit characteristic up to α_0 , and, so, S^0 on α_{τ_0} becomes unit matrix contrary to the supposition.

This argument mode consists in restrictions and extensions applied in turns and therefore we shall call it the *restriction-and-extension method*. It will be used further often enough in various forms and is typical in disseminator theory, therefore one should consider it more in details:

Let $\gamma < \gamma_{\tau^2}$ be an arbitrary cardinal; there exist the unit matrix S^0 on some carrier $\alpha > \gamma$ and it remains unit below α^0 due to lemma 8.7 about absoluteness. Now the reasoning passes to the situation below α^0 ; there holds the proposition below α^0 :

$$\exists \delta, \gamma_\tau, \alpha, \rho \left(\gamma < \gamma_\tau \wedge \alpha \mathbf{K}(1, \delta, \gamma_\tau, \alpha, \rho, S^0) \right),$$

it belongs to Σ_n and contains only constants

$$\chi^*, \gamma < \gamma_{\tau^2}, S^0 \triangleleft \chi^{*+} < \gamma_{\tau^2}.$$

Thus the Π_n -cardinal γ_{τ^2} below α_0 restricts it by lemma 3.2 [27], that is it holds after its bounding by γ_{τ^2} :

$$\exists \delta, \gamma_\tau^{<\gamma_{\tau^2}}, \alpha, \rho < \gamma_{\tau^2} \left(\gamma < \gamma_\tau^{<\gamma_{\tau^2}} \wedge \alpha \mathbf{K}^{\triangleleft \gamma_{\tau^2}}(1, \delta, \gamma_\tau^{<\gamma_{\tau^2}}, \alpha, \rho, S^0) \right).$$

But here the upper indices $< \gamma_{\tau^2}$, $\triangleleft \gamma_{\tau^2}$ can be dropped due to the $\Pi_n^{<\alpha^0}$ -subinaccessibility of γ_{τ^2} and as a result there appear admissible carriers

$$\alpha \in]\gamma, \gamma_{\tau^2}[$$

of the matrix S^0 of unit characteristic on such α below α^0 for arbitrary $\gamma < \gamma_{\tau^2}$.

Then by the inductive hypothesis $a_\tau \equiv 1$ on $] \tau_1, \tau^2[$, and below γ_{τ^2} there is fulfilled the proposition

$$\forall \tau \quad (\gamma_{\tau^3} < \gamma_\tau \longrightarrow a_\tau = 1)$$

that can be formulated in the Π_n -form:

$$\begin{aligned} \forall \gamma \Big(\gamma_{\tau^3} < \gamma \wedge SIN_{n-1}(\gamma) \longrightarrow \\ \longrightarrow \exists \delta, \alpha, \rho, S \Big(SIN_n^{<\alpha^\Downarrow}(\gamma_{\tau^3}) \wedge \alpha \mathbf{K}_{n+1}^{*\exists}(1, \delta, \gamma, \alpha, \rho, S) \Big) \Big) . \end{aligned}$$

The cardinal $\gamma_{\tau^2} \in SIN_n^{<\alpha^0}$ extends this last proposition up to α^0 , and below α^0 there appears the matrix of unit characteristic on some carrier $\in] \gamma_{\tau^0}, \alpha^0[$ admissible together with its disseminator $< \gamma_{\tau_0}$ and its base for γ_{τ_0} .

Thus, $a_{\tau_0} = 1$ contrary to the assumption, and we turn to the next subcase:

1b. $\check{\delta}^0 \in SIN_n$. Since $a_\tau \equiv 1$ on $[\tau^1, \tau^0[$, there exist the matrix

$$S^0 = \alpha S_{\tau^1,0} \text{ on the carrier } \alpha_{\tau^1,0} \in [\check{\delta}^0, \gamma_{\tau^0}[$$

of unit characteristic $a_{\tau^1,0} = 1$ and one can reveal the situation below $\alpha^{1,0} = \alpha_{\tau^1,0}^\Downarrow$ in the following way.

The reasoning forthcoming is applied further subsequently, thus it is necessary to dwell upon it.

We start with S^0 on α_{τ_0} . By lemma 8.5 5) zero characteristic of S^0 on α_{τ_0} means that there holds

$$\begin{aligned} \exists \tau'_1, \tau'_2, \tau'_3 < \alpha^0 \quad (A_2^{0<\alpha^0}(\tau'_1, \tau'_2, \tau'_3, \alpha S_f^{<\alpha^0}) \wedge \\ \wedge \forall \tau'' \in] \tau'_1, \tau'_2[\quad a_{\tau''}^{<\alpha^0} = 1 \wedge \alpha S_{\tau'_2}^{<\alpha^0} = S^0). \end{aligned} \tag{9.5}$$

Thus there can be used some ordinals

$$\tau'_1 < \tau'_2 < \tau'_3 < \alpha^0$$

such that there holds

$$A_2^0(\tau'_1, \tau'_2, \tau'_3, \alpha S_f) \wedge \forall \tau'' \in]\tau'_1, \tau'_2] \ a_{\tau''}^{<\alpha^0} = 1 \wedge \alpha S_{\tau'_2} = S^0 \quad (9.6)$$

below α^0 , that is after \triangleleft -bounding by the cardinal α^0 .

The key role will be played here by the so called *mediator*: it is some $SIN_n^{<\alpha^0}$ -cardinal γ^0 such that

$$\gamma_{\tau'_1}^{<\alpha^0} < \gamma_{\tau'_2}^{<\alpha^0} < \gamma_{\tau'_3}^{<\alpha^0} < \gamma^0 < \alpha^0 \quad (9.7)$$

which exist due to lemma 8.5 4). By lemma 8.7 about absoluteness of the admissibility and of unit values of the matrix α -function these values and their accessories below α^0 and below γ^0 coincide on the set

$$\{\tau : \gamma_{\tau}^{<\alpha^0} < \gamma^0 \wedge a_{\tau}^{<\alpha^0} = 1\}$$

and therefore (9.7), (9.6) imply the following Σ_{n+1} -proposition below α^0 :

$$\begin{aligned} \exists \gamma^0 \exists \tau'_1, \tau'_2, \tau'_3 < \gamma^0 (SIN_n(\gamma^0) \wedge \gamma_{\tau'_1}^{<\gamma^0} < \gamma_{\tau'_2}^{<\gamma^0} < \gamma_{\tau'_3}^{<\gamma^0} < \gamma^0 \wedge \\ \wedge A_2^{0\triangleleft\gamma^0}(\tau'_1, \tau'_2, \tau'_3, \alpha S_f^{<\gamma^0}) \wedge \forall \tau'' \in]\tau'_1, \tau'_2] \ a_{\tau''}^{<\gamma^0} = 1 \wedge \\ \wedge \alpha S_{\tau'_2}^{<\gamma^0} = S^0). \end{aligned} \quad (9.8)$$

Due to lemma 8.5 3)

$$\check{\delta}^0 \in SIN_{n+1}^{<\alpha^0}[<\rho_{\tau^0}]$$

and then by lemma 3.2 [27] there exist some γ^0 with the property (9.8), but *already below* $\check{\delta}^0$. From here and lemma 3.8 [27] it

follows that the SIN_n -subinaccessibility of $\check{\delta}^0$ draws the existence of SIN_n -cardinal $\gamma^{0'} < \check{\delta}^0$ with the same property (9.8); note, that $\gamma^{0'}$ possesses the same SIN_n -subinaccessibility as $\check{\delta}^0$. So, for some cardinals

$$\gamma_{\tau_1''}^{<\gamma^{0'}} < \gamma_{\tau_2''}^{<\gamma^{0'}} < \gamma_{\tau_3''}^{<\gamma^{0'}} < \gamma^{0'} \quad (9.9)$$

there holds

$$\begin{aligned} & A_2^{0 \triangleleft \gamma^{0'}}(\tau_1'', \tau_2'', \tau_3'', \alpha S_f^{<\gamma^{0'}}) \wedge \\ & \wedge \forall \tau''' \in]\tau_1'', \tau_2''] \ a_{\tau'''}^{<\gamma^{0'}} = 1 \wedge \alpha S_{\tau_2''}^{<\gamma^{0'}} = S^0 \end{aligned} \quad (9.10)$$

Since $\gamma^{0'}$ is the SIN_n -cardinal everywhere in (9.9), (9.10) \triangleleft - and $<$ -boundaries by $\gamma^{0'}$ can be dropped by the same lemmas 3.8, 8.7.

From this place one have to repeat the reasoning above but in the reserve direction, and not for α^0 , but for $\alpha^{1,0}$. Then (9.9), (9.10) draw (9.5) where α^0 is replaced with $\alpha^{1,0}$ and thereby S^0 on $\alpha_{\tau^{1,0}}$ receives *zero characteristic* contrary to the assumptions. 2. So, statement 1 is proved; it remains to examine the ordinal

$$\tau^{1,2} = \min\{\tau \in [\tau_1, \tau_2[: \forall \tau' \in]\tau, \tau_2[\ a_{\tau'} = 1\}$$

and to prove that it coincides with τ_1 .

Suppose it is wrong and $\tau_1 < \tau^{1,2}$, then one should examine two unit matrices

$$S^1 = \alpha S_{\tau^{1,2}}, \ S^2 = \alpha S_{\tau^{1,2}+1}$$

and treat the matrix S^2 on its carrier $\alpha_{\tau^{1,2}+1}$ with its generating disseminator $\check{\delta}^2 = \check{\delta}_{\tau^{1,2}+1}$. By lemmas 8.5 7) (ii), 8.8 2)

$$\gamma_{\tau_1} \leq \check{\delta}^2 = \check{\delta}_{\tau^{1,2}+1}$$

and cause of that only three subcases arises:

2a. $\gamma_{\tau_1} = \check{\delta}^2$. Then by definition 8.3

$$\forall \tau \in]\tau_1, \tau^{1,2}[\ a_\tau = 1$$

contrary to the supposition.

2b. $\gamma_{\tau_1} < \check{\delta}^2$, $\check{\delta}^2 \notin SIN_n$. Then the restriction-and-extension reasoning works, literally as it was in subcase 1a. of this proof above, but for

$$S^2, \check{\delta}^2 \text{ instead of } \alpha S_{\tau_0}, \check{\delta}^0$$

and again it comes $a_\tau \equiv 1$ on $]\tau_1, \tau^{1,2}[$.

2c. $\gamma_{\tau_1} < \check{\delta}^2$, $\check{\delta}^2 \in SIN_n$. Here again the restriction-and-extension method works, but in slightly another manner. First by lemma 3.2 [27] matrix S^1 receives its admissible carriers of unit characteristic disposed cofinally to $\check{\delta}^2$, so by the inductive hypothesis

$$a_\tau \equiv 1 \text{ on the set } \{\tau : \gamma_{\tau_1} < \gamma_\tau < \check{\delta}^2\}.$$

Then below $\check{\delta}^2$ the following Π_{n+1} -proposition holds

$$\forall \gamma (\gamma_{\tau_1} < \gamma \wedge SIN_{n-1}(\gamma) \rightarrow \exists \delta, \alpha, \rho, S \ \alpha \mathbf{K}(1, \delta, \alpha, \gamma, \rho, S))$$

which is extended by this disseminator up to $\alpha_{\tau^{1,2}+1}^\Downarrow$ according to lemma 6.6 [27] (for $m = n + 1$, $\delta = \check{\delta}^2$, $\alpha_1 = \alpha_{\tau^{1,2}+1}^\Downarrow$) and again it comes $a_\tau \equiv 1$ on the same set $]\tau_1, \tau^{1,2}[$.

In every case it implies $\tau_1 = \tau^{1,2}$. \dashv

The following important lemma will be proved again by means of the *restriction-and-extension method* but in some synthesized form.

However, beforehand the following rather suitable notion should be introduced using the notions of reduced spectra and matrices (remind definitions 4.1, 5.1 [27]).

In what follows the main technical mode of reasonings will consist in the examination of some matrix S under consideration on its *different carriers in turns*. Such transition of the reduced matrix S from one its carrier α over to another its carrier α^1 will be called the *carrying over* of the matrix S from α to α^1 .

This technique will be frequent enough to be used further and was already used above in the proofs of lemmas 7.5, 8.8, 9.4.

During such carrying over of reduced matrix S from α to α^1 some properties of the universe bounded by jump or prejump cardinals of S on α can be preserved and thereby they will be called the *inner properties* of S ; other properties of S may be lost and they will be called the *outer properties*.

More precisely: a property or attribute of matrix S reduced to χ^* on its carrier α will be called the *inner* property or attribute of this S (on α) if it is definable below some jump or prejump cardinal of the spectrum

$$\text{dom}(\tilde{\mathbf{S}}_n^{\sin \triangleleft \alpha} \upharpoonright \chi^*)$$

through its some other jump or prejump cardinals; analogously for other objects from L_α ; in all other cases they will be called the *outer properties or attributes or objects* of S .

These notions are activated by lemma 5.11 [27] about matrix informativeness which means that such *inner* properties are preserved while matrix S is carried over from one its carrier to any other one.

Here is very important example of the *outer* property – the property of *characteristic*; it involves the *whole* matrix S on its carrier α , but not only its some jump cardinals.

Really, take any matrix S on its carrier α of *zero* characteristic (if such exist), then by lemma 8.5 5) there holds

$$\begin{aligned} \exists \tau'_1, \tau'_2, \tau'_3 < \alpha^\Downarrow (A_2^{0\triangleleft \alpha^\Downarrow}(\tau'_1, \tau'_2, \tau'_3, \alpha S_f^{< \alpha^\Downarrow}) \wedge \\ \wedge \forall \tau'' \in]\tau'_1, \tau'_2] \ a_{\tau''}^{< \alpha^\Downarrow} = 1 \wedge \alpha S_{\tau'_2}^{< \alpha^\Downarrow} = S), \end{aligned}$$

where S receives the *lesser* carrier $\alpha_{\tau'_2}^{< \alpha^\Downarrow}$, already of the *unit* characteristic due to the condition $\forall \tau'' \in]\tau'_1, \tau'_2] \ a_{\tau''}^{< \alpha^\Downarrow} = 1$.

But other matrix properties used in what follows are *inner*, and one of them realizes the restriction-and-extension reasoning in the following lemma.

This lemma uses the suitable function, which was already used in the proof of lemma 9.3:

$$OdaS_f^{< \alpha_1}(\tau_1, \tau_2) = \sup\{Od(\alpha S_\tau^{< \alpha_1}) : \tau_1 < \tau < \tau_2\};$$

it will be applied to forming the so called *stairways* — collections of intervals, which will be the main technical tools in the Main theorem proof. To this end the following formulas below α_1 are needed:

$$1. \ A_{1.1}^{m\triangleleft \alpha_1}(\tau_1, \tau_2, \alpha S_f^{< \alpha_1}):$$

$$A_1^{1\triangleleft \alpha_1}(\tau_1, \tau_2, \alpha S_f^{< \alpha_1}) \wedge \tau_2 = \sup\{\tau : A_1^{1\triangleleft \alpha_1}(\tau_1, \tau_2, \alpha S_f^{< \alpha_1})\};$$

here the interval $[\tau_1, \tau_2[$ is the maximal of monotonicity intervals with the left $SIN_n^{< \alpha_1}$ -end $\gamma_{\tau_1}^{< \alpha_1}$ and with right $SIN_n^{< \alpha_1}$ -ends, thus we shall call it and the corresponding interval $[\gamma_{\tau_1}^{< \alpha_1}, \gamma_{\tau_2}^{< \alpha_1}[$ the *maximal intervals* of the function $\alpha S_f^{< \alpha_1}$ *monotonicity* below α_1 .

$$2. \ A_{1.1}^{m1\triangleleft \alpha_1}(\tau_1, \tau_2, \alpha S_f^{< \alpha_1}, a_f^{< \alpha_1}):$$

$$A^{0\triangleleft \alpha_1}(\tau_1) \wedge A_{1.1}^{m\triangleleft \alpha_1}(\tau_1, \tau_2, \alpha S_f^{< \alpha_1}) \wedge \forall \tau (\tau_1 < \tau < \tau_2 \rightarrow a_\tau^{< \alpha_1} = 1);$$

in addition to $A_{1.1}^{m\triangleleft \alpha_1}$ here is stated, that there is no α -matrices

admissible for $\gamma_{\tau_1}^{<\alpha_1}$ below α_1 and the function $\alpha S_f^{<\alpha_1}$ has on $]\tau_1, \tau_2[$ the values $\alpha S_{\tau}^{<\alpha_1}$ only of unit characteristic $a_{\tau}^{<\alpha_1} = 1$; in such cases the unit characteristic $a = 1$ stabilizes on the interval $[\tau_1, \tau_2[$ and on the corresponding interval $[\gamma_{\tau_1}^{<\alpha_1}, \gamma_{\tau_2}^{<\alpha_1}[$ below α_1 by lemma 9.4

$$3. \quad A_{1.1}^{st \triangleleft \alpha_1}(\tau_1, \tau_*, \tau_2, \alpha S_f^{<\alpha_1}, a_f^{<\alpha_1}):$$

$$A_{1.1}^{m1 \triangleleft \alpha_1}(\tau_1, \tau_*, \alpha S_f^{<\alpha_1}, a_f^{<\alpha_1}) \wedge \tau_1 < \tau_* \leq \tau_2 \wedge A_1^{\triangleleft \alpha_1}(\tau_1, \tau_2, \alpha S_f^{<\alpha_1});$$

here is indicated, that the function $\alpha S_f^{<\alpha_1}$ is defined on the interval $]\tau_1, \tau_2[$, but on its maximal initial subinterval of monotonicity $]\tau_1, \tau_*[$ with $\gamma_{\tau_*} \in SIN_n^{<\alpha_1}$ it has even the *unit* characteristic stabilized on it; therefore the interval $[\tau_1, \tau_2[$ and the corresponding interval $[\gamma_{\tau_1}^{<\alpha_1}, \gamma_{\tau_2}^{<\alpha_1}[$ will be called further the (unit) steps below α_1 ; in this case the ordinal

$$Od \alpha S_f^{<\alpha_1}(\tau_1, \tau_*)$$

will be called the *height* of this step.

$$4. \quad A_{1.1}^{Mst \triangleleft \alpha_1}(\tau_1, \tau_*, \tau_2, \alpha S_f^{<\alpha_1}, a_f^{<\alpha_1}):$$

$$A_{1.1}^{st \triangleleft \alpha_1}(\tau_1, \tau_*, \tau_2, \alpha S_f^{<\alpha_1}, a_f^{<\alpha_1}) \wedge A_{1.1}^{M \triangleleft \alpha_1}(\tau_1, \tau_2, \alpha S_f^{<\alpha_1});$$

in addition here is indicated, that the interval $]\tau_1, \tau_2[$ is the maximal with $\gamma_{\tau_2}^{<\alpha_1} \in SIN_n^{<\alpha_1}$, thereby we shall call the interval $[\tau_1, \tau_2[$ and the corresponding interval $[\gamma_{\tau_1}^{<\alpha_1}, \gamma_{\tau_2}^{<\alpha_1}[$ the *maximal (unit) steps* below α_1 .

This survey leads to the notion of stairway:

$$5. \quad A_8^{St \triangleleft \alpha_1}(\mathcal{St}, \chi, \alpha S_f^{<\alpha_1}, a_f^{<\alpha_1}):$$

$$(\mathcal{St} - \text{is a function on } \chi^{*+}) \wedge$$

$$\begin{aligned}
& \wedge \forall \beta < \chi^{*+} \exists \tau_1, \tau_*, \tau_2 (\mathcal{St}(\beta) = (\tau_1, \tau_*, \tau_2) \wedge \\
& \quad \wedge A_{1.1}^{Mst < \alpha_1}(\tau_1, \tau_*, \tau_2, \alpha S_f^{< \alpha_1}, a_f^{< \alpha_1}) \wedge \\
& \quad \wedge \forall \tau_1, \tau_*, \tau_2 (A_{1.1}^{Mst < \alpha_1}(\tau_1, \tau_*, \tau_2, \alpha S_f^{< \alpha_1}, a_f^{< \alpha_1}) \longrightarrow \\
& \quad \longrightarrow \exists \beta < \chi^{*+} \mathcal{St}(\beta) = (\tau_1, \tau_*, \tau_2)) \wedge \\
& \quad \wedge \forall \beta_1, \beta_2 < \chi^{*+} \forall \tau'_1, \tau'_*, \tau'_2 \forall \tau''_1, \tau''_*, \tau''_2 (\beta_1 < \beta_2 \wedge \\
& \quad \wedge \mathcal{St}(\beta_1) = (\tau'_1, \tau'_*, \tau'_2) \wedge \mathcal{St}(\beta_2) = (\tau''_1, \tau''_*, \tau''_2) \rightarrow \tau'_2 < \tau''_1 \wedge \\
& \quad \wedge Od \alpha S_f^{< \alpha_1}(\tau'_1, \tau'_*) < Od \alpha S_f^{< \alpha_1}(\tau''_1, \tau''_*) \wedge \\
& \quad \wedge \sup \{Od \alpha S_f^{< \alpha_1}(\tau_1, \tau_*) : \exists \beta, \tau_2 \mathcal{St}(\beta) = (\tau_1, \tau_*, \tau_2)\} = \chi^{*+};
\end{aligned}$$

here is indicated, that \mathcal{St} is the function on χ^{*+} , and its values are all triples (τ_1, τ_*, τ_2) such that the intervals $[\tau_1, \tau_2[$ are maximal unit steps disposed successively one after another. Therefore such \mathcal{St} will be called the *stairway* and the intervals $[\tau_1, \tau_2[$ and the corresponding intervals $[\gamma_{\tau_1}^{< \alpha_1}, \gamma_{\tau_2}^{< \alpha_1}[$ – its steps below α_1 .

This notion is justified by the strict increasing of their heights; also we shall say, that the stairway \mathcal{St} consists of these steps, or contains them.

Respectively, the cardinal

$$h(\mathcal{St}) = \sup \{Od \alpha S_f^{< \alpha_1}(\tau_1, \tau_*) : \exists \beta, \tau_2 \mathcal{St}(\beta) = (\tau_1, \tau_*, \tau_2)\}$$

will be called the *height* of the whole stairway \mathcal{St} . So, here is required that \mathcal{St} amounts up to χ^{*+} , that is

$$h(\mathcal{St}) = \chi^{*+}.$$

Also the cardinal

$$v = \sup \{\gamma_{\tau_2} : \exists \beta, \tau_1, \tau_* \mathcal{St}(\beta) = (\tau_1, \tau_*, \tau_2)\}$$

will be called the termination cardinal of $\mathcal{S}t$ and will be denoted through

$$v(\mathcal{S}t);$$

so, we shall say, that the stairway $\mathcal{S}t$ terminates in this cardinal $v(\mathcal{S}t)$.

If such stairway $\mathcal{S}t$ exist below α_1 , then we shall say, that α_1 is *provided* by this stairway $\mathcal{S}t$.

When $\alpha > \chi^*$ is a carrier of the matrix S and its prejump cardinal $\alpha_1 = \alpha_\chi^\downarrow$ after χ^* is provided by some stairway $\mathcal{S}t$, then we shall say, that this S on α is provided by this stairway.

And here is quite important example of the inner phenomena: the *inner* property of providing the matrix S by some stairway. This property for S on its carrier α is definable by the formula

$$\exists \mathcal{S}t \triangleleft \alpha^{\downarrow+} A_8^{\mathcal{S}t \triangleleft \alpha^{\downarrow}}(\mathcal{S}t, \alpha S_f^{< \alpha^{\downarrow}}, a_f^{< \alpha^{\downarrow}})$$

which can be bounded by the jump cardinal α^\downarrow of the carrier α after χ^* . Therefore by lemma 5.11 [27] the same property holds for S on any other carrier $\alpha^1 > \chi^*$:

$$\exists \mathcal{S}t^1 \triangleleft \alpha^{1\downarrow+} A_8^{\mathcal{S}t^1 \triangleleft \alpha^{1\downarrow}}(\mathcal{S}t^1, \alpha S_f^{< \alpha^{1\downarrow}}, a_f^{< \alpha^{1\downarrow}})$$

being bounded by the jump cardinal $\alpha^{1\downarrow}$ of α^1 after χ^* , and, so, S on α^1 is again provided by some stairway $\mathcal{S}t^1$ as well.

Lemma 9.5 (About stairway cut-off from above)

Let

$$(i) \quad A_1^{1 \triangleleft \alpha_1}(\tau_1, \tau_2);$$

(ii) $\tau_2 \leq \tau_3$ and S^3 be a matrix of characteristic α^3 on a carrier

$$\alpha_3 \in]\gamma_{\tau_3}^{< \alpha_1}, \alpha_1[$$

with disseminator $\tilde{\delta}^3$ and data base ρ^3 admissible for $\gamma_{\tau_3}^{<\alpha_1}$ below α_1 and with the generating eigendisseminator $\check{\delta}^{S^3}$ on α^3 ;

$$(iii) \quad \forall \tau < \tau_2 \quad \exists \tau' \in [\tau, \tau_2[\quad a_{\tau'}^{<\alpha_1} = a^3.$$

Then

$$1. \quad Od\alpha S_f^{<\alpha_1}(\tau_1, \tau_2) < Od(S^3);$$

2a. hence, if $a^3 = 1$, then there is no stairway below α_1 terminating in some $SIN_n^{<\alpha_1}$ -cardinal $v < \alpha_3^\downarrow$;

2b. therefore if there exist some unit matrix S^0 on its carriers admissible below α_1 and disposed cofinally to α_1 :

$$\begin{aligned} \forall \gamma < \alpha_1 \quad \exists \gamma^1 \in]\gamma, \alpha_1[\quad \exists \delta, \alpha, \rho < \alpha_1 \quad (SIN_{n-1}^{<\alpha_1}(\gamma^1) \wedge \\ \wedge \alpha \mathbf{K}^{<\alpha_1}(1, \delta, \gamma^1, \alpha, \rho, S^0)), \end{aligned}$$

then α_1 is not provided by any stairway;

3. if S^3 is the \leq -minimal of all matrices of the same characteristic a^3 on carriers $\in]\gamma_{\tau_3}^{<\alpha_1}, \alpha_1[$ admissible for $\gamma_{\tau_3}^{<\alpha_1}$, then

$$\gamma_{\tau_2}^{<\alpha_1} < \check{\delta}^{S^3} \leq \tilde{\delta}^3 < \gamma_{\tau_3}.$$

Proof. Let us demonstrate the reasoning for the case $a^3 = 1$, used in what follows; in this important case $\tilde{\delta}^3 = \check{\delta}^{S^3}$ and condition (iii) should be weakened up to $a^3 = 1$ by lemma 3.2 [27]. In this case the nonsuppression condition $\neg A_5^{S,0}$ for the unit matrix S on its carriers can be dropped, because such S is always nonsuppressed and the formulas $\alpha \mathbf{K}^{*<\alpha_1}, \alpha \mathbf{K}_{n+1}^{*<\alpha_1}$ are equivalent to the formulas $\alpha \mathbf{K}^{<\alpha_1}, \alpha \mathbf{K}_{n+1}^{<\alpha_1}$; the upper indices $< \alpha_1, \triangleleft \alpha_1$ will be omitted.

By this lemma the matrix S^3 receives unit characteristic on its admissible carriers disposed cofinally to γ_{τ_2} , as it was in part 1a. in lemma 9.4 proof, where γ_{τ_2} , S^0 on α_{τ_1} should be replaced with γ_{τ_2} , S^3 on α_3 . So (i) and lemma 9.4 imply

$$Od\alpha S_f(\tau_1, \tau_2) \leq Od(S^3) \text{ and } a_\tau \equiv 1 \text{ on }]\tau_1, \tau_2[.$$

Now let us assume that the function αS_f stabilizes on $[\tau_1, \tau_2[$ and let τ_2^s be the stabilization ordinal of αS_f on $[\tau_1, \tau_2[$ so that there exists S^0 such that

$$\alpha S_\tau \equiv \alpha S_{\tau_2^s} = S^0 \text{ on } [\tau_2^s, \tau_2[.$$

We apply now the restriction-and-extension argument mode, that was used several times above. Let us turn to the matrix S^0 on the carrier $\alpha_{\tau_2^s+1}$ with the prejump cardinal $\alpha^1 = \alpha_{\tau_2^s+1}^\downarrow$ and the disseminator $\check{\delta}^1 = \check{\delta}_{\tau_2^s+1}$. The same matrix S^0 on the carrier $\alpha_{\tau_2^s}$ of unit characteristic by lemma 3.2 [27] about restriction receives unit characteristic also on its admissible carriers disposed cofinally to $\check{\delta}^1$ and hence below $\check{\delta}^1$ the following Π_{n+1} -proposition holds for $S = S^0$:

$$\begin{aligned} \forall \gamma \exists \gamma' (\gamma < \gamma' \wedge SIN_{n-1}(\gamma') \wedge \\ \wedge \exists \delta, \alpha, \rho \ \alpha \mathbf{K}(1, \delta, \gamma', \alpha, \rho, S)) . \end{aligned} \quad (9.11)$$

Disseminator $\check{\delta}^1$ extends it up to α^1 and so the matrix S^0 receives unit characteristic on its admissible carriers disposed cofinally to α^1 , that is (9.11) is fulfilled by $S = S^0$ under the boundary $\triangleleft \alpha^1$. After the minimizing such matrices S we receive the matrix $S = S^1$ with the property (9.11) below α^1 and by lemma 4.6 [27] $S^1 \triangleleft S^0$. One should point out, that statement (9.11) \triangleleft -bounded by α^1 with $S = S^1$ is the *inner property* of the matrix S^0 .

If now

$$Od\alpha S_f(\tau_1, \tau_2) = Od(S^3), \text{ that is } S^0 = S^3,$$

then matrix S^1 by lemma 5.11 [27] about informativeness receives its admissible carriers with the same unit characteristic disposed cofinally to the prejump cardinal $\alpha^3 = \alpha_3^\Downarrow$, because S^0 on the carrier $\alpha_{\tau_2^s+1}$ has the same property.

After that again by lemma 3.2 [27] such carriers appear disposed cofinally to γ_{τ_2} . Hence, at last, by (i) it comes the contradiction:

$$Od\alpha S_f(\tau_1, \tau_2) \leq Od(S^1) < Od(S^0). \quad (9.12)$$

If the function αS_f does not stabilize on $[\tau_1, \tau_2[$ then the ordinal

$$\rho = Od\alpha S_f(\tau_1, \tau_2)$$

is limit. But here to finish the proof of 1. one should remind, that Gödel function F has values $F_\alpha = F|_\alpha$ for any limit ordinals α (see Gödel [22]). It is not hard to see that $Od(S)$ can not be limit and thus $\rho \leq Od(S)$ implies $\rho < Od(S)$.

Turning to 3. let us suppose that it is wrong and

$$\check{\delta}^3 = \check{\delta}^{S^3} \leq \gamma_{\tau_2}$$

and standing on $\alpha^3 = \alpha_3^\Downarrow$ let us review the situation below α^3 obtained. Two cases here should be considered:

1. $\gamma_{\tau_1} < \check{\delta}^3 \leq \gamma_{\tau_2}$.

Since $\gamma_{\tau_2} \in SIN_n$ and $\check{\delta}^3 \in SIN_n^{<\alpha^3}$ lemmas 3.8 [27], 8.5 1), 8.7 imply that $\check{\delta}^3 \in SIN_n$ and

$$\alpha S_\tau^{<\alpha^3} \equiv \alpha S_\tau \quad \text{on} \quad \{\tau : \gamma_{\tau_1} < \gamma_\tau < \check{\delta}^3\}.$$

The disseminator $\check{\delta}^3$ extends up to α^3 the Π_{n+1} -proposition stating the definiteness of the function αS_f of *unit* characteristic

and with values $\prec S^3$ due to part 1.:

$$\begin{aligned} \forall \gamma' \left(\gamma_{\tau_1} < \gamma' \wedge SIN_{n-1}(\gamma') \longrightarrow \right. \\ \left. \longrightarrow \exists \delta, \alpha, \rho, S \left(S \prec S^3 \wedge \alpha \mathbf{K}(1, \delta, \gamma', \alpha, \rho, S) \right) \right) \end{aligned} \quad (9.13)$$

and, hence, there exists the matrix $\alpha S_{\tau_3}^{<\alpha^3}$ on the carrier

$$\alpha_{\tau_3}^{<\alpha^3} \in]\gamma_{\tau_3}, \alpha^3[$$

of unit characteristic and $\alpha S_{\tau_3}^{<\alpha^3} \prec S^3$ contrary to the \leq -minimality of S^3 on α_3 . It remains to consider the case:

2. $\check{\delta}^3 \leq \gamma_{\tau_1}$.

It should be pointed out that the condition of \leq -minimality of S^3 is not used in this case. Here the matrix $S^0 = \alpha S_{\tau_1+2}$ of unit characteristic should be considered on its carrier α_{τ_1+2} with the pre-jump cardinal $\alpha^1 = \alpha_{\tau_1+2}^{\Downarrow}$ and the disseminator $\check{\delta}^1 = \check{\delta}_{\tau_1+2}$, just as it was done above for $S^0 = \alpha S_{\tau_2^s+1}$, $\alpha^1 = \alpha_{\tau_2^s+1}^{\Downarrow}$, $\check{\delta}^1 = \check{\delta}_{\tau_2^s+1}$ in the proof of 1. (let us preserve the previous notations for convenience). And again the matrix αS_{τ_1+1} receives the unit characteristic on its admissible carriers disposed cofinally to the disseminator $\check{\delta}^1 = \gamma_{\tau_1}$ and it extends proposition (9.11) for $S = \alpha S_{\tau_1+1}$ up to α^1 ; so, it brings by the same way the minimal matrix $S^1 \prec S^0$ with the previous properties: it receives unit characteristic on its admissible carriers disposed cofinally to α^1 .

By lemma 3.2 [27] there appear carriers of S^1 of unit characteristic disposed cofinally to $\check{\delta}^3$, that is (9.11) is fulfilled by $S = S^1$ under the \prec -bounding by $\check{\delta}^3$; hence, the disseminator $\check{\delta}^3$ extends this proposition up to α^3 . After that the cardinal $\gamma_{\tau_2} \in SIN_n$ restricts this proposition with γ substituted for an arbitrary constant $\gamma^1 < \gamma_{\tau_2}$. As a result the matrix S^1 receives the unit

characteristic on its admissible carriers disposed cofinally to γ_{τ_2} and so again we come to contradiction (9.12).

Turning to $\alpha^3 = 0$ one should repeat all this proof but for *zero* matrices S on their carriers α admissible for cardinals γ_τ under consideration, but only for $\gamma_\tau \notin SIN_n$. In all cases of this kind such matrices S are nonsuppressed by definition and again the nonsuppression condition $\neg A_5^{S,0}$ can be dropped, and the formulas $\alpha\mathbf{K}^{*<\alpha_1}$, $\alpha\mathbf{K}_{n+1}^{*<\alpha_1}$ can be replaced with the formulas $\alpha\mathbf{K}^{<\alpha_1}$, $\alpha\mathbf{K}_{n+1}^{<\alpha_1}$. Precisely such matrices S on their carriers α can be used in the restriction-and-extension reasoning above, that provides the proof for zero characteristic $a_3 = 0$.

At last proposition 2a. comes from 1. almost obviously. Suppose it fails, that is there exist some stairway \mathcal{St} below α_1 terminating in the SIN_n -cardinal $v(\mathcal{St}) < \alpha_3^\downarrow$; it implies that

$$v(\mathcal{St}) < \gamma_{\tau_3}^{<\alpha_1}.$$

By definition this stairway consists of *unit* steps below α_1

$$\mathcal{St}(\tau') = (\tau'_1, \tau'_2, \tau'_2)$$

and each of them possesses the property

$$A_1^{1<\alpha_1}(\tau'_1, \tau'_2)$$

with the unit characteristic stabilizing on $[\tau'_1, \tau'_2[$ (see definition of the stairway notion before lemma 9.5). Hence 1. provides the height $h(\mathcal{St})$ of the whole \mathcal{St} bounded by the ordinal

$$Od(S^3) < \chi^{*+}$$

though $h(\mathcal{St}) = \chi^{*+}$, that is \mathcal{St} amounts up to χ^{*+} by definition.

From here it follows 2b. when the matrix S^0 is used instead of the matrix S^3 . \dashv

One should notice, that for $a^3 = 1$ the condition of the matrix S^3 minimality in point 3. of this lemma 9.5 can be dropped by means of the reasoning repeating the arguments above in case 1. slightly changed.

Next obvious corollary shows that such steps heights always increase strictly:

Corollary 9.6

Let

$$(i) \quad A_1^{1 \triangleleft \alpha_1}(\tau_1, \tau_2), \quad A_1^{1 \triangleleft \alpha_1}(\tau_3, \tau_4), \quad \tau_2 < \tau_4;$$

$$(ii) \quad \forall \tau < \tau_4 \quad \exists \tau' \in [\tau; \tau_4[\quad a_{\tau'}^{< \alpha_1} = 1.$$

Then

$$1) \quad \forall \tau \in]\tau_1, \tau_2[\cup]\tau_3, \tau_4[\quad a_{\tau}^{< \alpha_1} = 1;$$

$$2) \quad Od\alpha S_f^{< \alpha_1}(\tau_1, \tau_2) < Od\alpha S_f^{< \alpha_1}(\tau_3, \tau_4);$$

$$3) \quad \forall \tau \in]\tau_3, \tau_4[\quad \gamma_{\tau_2}^{< \alpha_1} < \check{\delta}_{\tau}^S = \tilde{\delta}_{\tau}^{< \alpha_1} < \gamma_{\tau}^{< \alpha_1},$$

where $\check{\delta}_{\tau}^S$ is the generating eigendissemimator of $\alpha S_{\tau}^{< \alpha_1}$ on $\alpha_{\tau}^{< \alpha_1}$; hence

$$\gamma_{\tau_2}^{< \alpha_1} < \gamma_{\tau_3}^{< \alpha_1}.$$

Proof. From conditions (i), (ii) and lemmas 3.2 [27], 9.4 there follows that $a_{\tau}^{< \alpha_1} \equiv 1$ on the intervals $] \tau_1, \tau_2[$, $] \tau_3, \tau_4[$. Therefore lemma 9.5 (where τ_3 plays the role of any $\tau \in] \tau_3, \tau_4[$) implies statements 2), 3). For $\tau = \tau_3 + 1$ here it comes $\gamma_{\tau_2}^{< \alpha_1} < \check{\delta}_{\tau}^{< \alpha_1}$ and at the same time by lemmas 8.5 7) (ii), 8.8 2) – $\check{\delta}_{\tau}^S = \gamma_{\tau_3}^{< \alpha_1}$; so, $\tau_2 < \tau_3$.

—

Corollary 9.7*Let*

$$(i) \quad A_1^{1 \triangleleft \alpha_1}(\tau_1, \tau_2);$$

$$(ii) \quad \tau_3 \in \text{dom}(\alpha S_f^{< \alpha_1}), \quad \tau_3 \geq \tau_2;$$

(iii) a matrix $\alpha S_{\tau_3}^{< \alpha_1}$ on $\alpha_{\tau_3}^{< \alpha_1}$ has generating eigendisseminator

$$\check{\delta}_{\tau_3}^S \leq \gamma_{\tau_2}^{< \alpha_1}$$

below α_1 .*Then*

$$1) \quad a_{\tau}^{< \alpha_1} \equiv 1 \quad \text{on} \quad]\tau_1, \tau_2[, \quad a_{\tau_3}^{< \alpha_1} = 0;$$

$$2) \quad \check{\delta}_{\tau_3}^S \leq \gamma_{\tau_1}^{< \alpha_1} \quad \text{and}$$

$$3) \quad \text{Od} \alpha S_f^{< \alpha_1}(\tau_1, \tau_2) > \text{Od}(\alpha S_{\tau_3}^{< \alpha_1}).$$

Analogously for the generating disseminator $\check{\delta}_{\tau_3}^{< \alpha_1}$ of $\alpha S_{\tau_3}^{< \alpha_1}$ on $\alpha_{\tau_3}^{< \alpha_1}$.

Proof. We shall omit the upper indices $< \alpha_1, \triangleleft \alpha_1$. By lemma 9.5 for $S^3 = \alpha S_{\tau_3}^{< \alpha_1}$ condition (iii) implies that for some $\tau < \tau_2$

$$\forall \tau' \in [\tau, \tau_2[\quad a_{\tau'} \neq a_{\tau_3};$$

due to lemma 3.2 [27] it is possible only when

$$\forall \tau' \in [\tau, \tau_2[\quad a_{\tau'} = 1, \quad a_{\tau_3} = 0$$

and then by lemma 9.4 $a_{\tau} \equiv 1$ on $] \tau_1, \tau_2[$.

If $\check{\delta}_{\tau_3}^S \in] \gamma_{\tau_1}, \gamma_{\tau_2}[$, then one can obtain $a_{\tau_3} = 1$ again using the reasoning from the proof of lemma 9.5, and extending proposition (9.13) without its subformula $S \triangleleft S^3$ by the disseminator

$\check{\delta}_{\tau_3}^S$ up to $\alpha_{\tau_3}^\Downarrow$, that provides $a_{\tau_3}^{<\alpha_1} = 1$.

After that it is enough to conduct the reasoning from the end of this proof (case 2.) repeated literally by means of restriction-and-extension method.

—

The immediate consequence of this lemma for $\tau_2 = \tau_3$ is the following

Theorem 1.

Let

(i) $\alpha S_f^{<\alpha_1}$ be monotone on $[\tau_1, \tau_2[$ below α_1 ;

(ii) $\tau_1 = \min\{\tau :]\tau, \tau_2[\subseteq \text{dom}(\alpha S_f^{<\alpha_1})\}$.

Then

$$]\gamma_{\tau_1}^{<\alpha_1}, \gamma_{\tau_2}^{<\alpha_1}[\cap SIN_n^{<\alpha_1} = \emptyset.$$

Proof. Let us suppose that, on the contrary, there exists $SIN_n^{<\alpha_1}$ -cardinal $\gamma_{\tau'_2}^{<\alpha_1} \in]\gamma_{\tau_1}^{<\alpha_1}, \gamma_{\tau_2}^{<\alpha_1}[$.

Then $\gamma_{\tau_1}^{<\alpha_1}$ belongs to $SIN_n^{<\alpha_1}$ as well; one can see it repeating literally the argument from the proof of lemma 8.10. So, the statement $A_1^{1<\alpha_1}(\tau_1, \tau'_2)$ holds on; it remains to apply corollary 9.7 using τ'_2 as $\tau_2 = \tau_3$, since $\check{\delta}_{\tau'_2}^S < \gamma_{\tau'_2}^{<\alpha_1}$ by definition.

—

10 Analysis of α -Function Nonmonotonicity

So, any interval of the α -function monotonicity can not be “too long” by theorem 1.

However, such function can be defined on “rather long” intervals; for example, the function $\alpha S_f^{<\alpha_1}$ is defined on the final segment T^{α_1} of any sufficiently great SIN_n -cardinal $\alpha_1 < k$ (lemma 8.9). Hence, its monotonicity on this segment is violated on some ordinals.

How does this phenomenon happen? In this section all substantial violations of this kind are analyzed. To this end the formula $A_2^{<\alpha_1}(\tau_1, \tau_2, \tau_3)$ should be recalled (see definition 8.1 1.4 for $X_1 = \alpha S_f^{<\alpha_1}$):

$$A_1^{<\alpha_1}(\tau_1, \tau_3, \alpha S_f^{<\alpha_1}) \wedge \tau_1 + 1 < \tau_2 < \tau_3 \wedge$$

$$\wedge \tau_2 = \sup\{\tau < \tau_3 : \forall \tau', \tau'' (\tau_1 < \tau' < \tau'' < \tau \rightarrow \alpha S_{\tau'}^{<\alpha_1} \leq \alpha S_{\tau''}^{<\alpha_1})\}.$$

So, here τ_2 is the *minimal* ordinal breaking the monotonicity of the function $\alpha S_f^{<\alpha_1}$ on the interval $[\tau_1, \tau_3[$. Thus, in all reasonings of this paragraph some nonmonotonicity $A_2^{<\alpha_1}(\tau_1, \tau_2, \tau_3)$ on the intervals $[\tau_1, \tau_3[$ is treated in different situations (but the condition $SIN_n^{<\alpha_1}(\gamma_{\tau_3})$ can be dropped everywhere except the last lemma 10.5).

Lemma 10.1

Let

$$(i) \quad A_2^{<\alpha_1}(\tau_1, \tau_2, \tau_3);$$

$$(ii) \quad SIN_n^{<\alpha_{\tau_2}^{\downarrow}} \cap \gamma_{\tau_2}^{<\alpha_1} \subseteq SIN_n^{<\alpha_1}.$$

Then

$$1) \ a_{\tau}^{<\alpha_1} \equiv 1 \text{ on }]\tau_1, \tau_2[, \ a_{\tau_2}^{<\alpha_1} = 0 \quad \text{and}$$

$$2) \ \tilde{\delta}_{\tau_2}^{<\alpha_1} \leq \gamma_{\tau_1}^{<\alpha_1}. \quad 13)$$

Proof. The upper indices $<\alpha_1, \triangleleft \alpha_1$ will be dropped. Since the function αS_f is monotone on $] \tau_1, \tau_2[$, from theorem 1 it comes

$$]\gamma_{\tau_1}, \gamma_{\tau_2}[\cap SIN_n = \emptyset. \quad (10.1)$$

Standing on $\alpha^2 = \alpha_{\tau_2}^{\Downarrow}$, let us consider below α^2 the function $\alpha S_f^{<\alpha^2}$. By (ii) and lemma 8.7 about absoluteness it coincides with αS_f on $] \tau_1, \tau_2[$ and is monotone on this interval.

That is why $\tilde{\delta}_{\tau_2} \leq \gamma_{\tau_1}$, otherwise $\tilde{\delta}_{\tau_2} \in]\gamma_{\tau_1}, \gamma_{\tau_2}[$ contrary to (ii), (10.1).

If now $a_{\tau_2} = 1$, then by lemma 9.2

$$Od\alpha S_f(\tau_1, \tau_2) \leq Od(\alpha S_{\tau_2})$$

in spite of (i) and so $a_{\tau_2} = 0$. The same happens if

$$\forall \tau < \tau_2 \quad \exists \tau' \in [\tau; \tau_2[\quad a_{\tau'} = 0$$

because in this case due to condition (i) one can consider $\tau_1^2 \in]\tau_1, \tau_2[$ such that for $S^2 = \alpha S_{\tau_2}$

$$a_{\tau_1^2} = 0, \quad \alpha S_{\tau_1^2} \succ S^2. \quad (10.2)$$

Due to consequence 2) and lemma 3.2 [27] about restriction *zero* matrix S^2 receives some admissible carrier $\alpha \in]\gamma_{\tau_1^2}, \gamma_{\tau_1^2+1}[$ as a result of restricting by SIN_{n-1} -cardinal $\gamma_{\tau_1^2+1}$ of the following Σ_{n-1} -proposition

$$\begin{aligned} \exists \alpha \Big(\gamma_{\tau_1^2} < \alpha \wedge \exists \delta, \alpha, \rho \Big(\delta \leq \gamma_{\tau_1} \wedge SIN_n^{<\alpha^{\Downarrow}}(\gamma_{\tau_1}) \wedge \\ \wedge \alpha \mathbf{K}_{n+1}^{\exists}(0, \delta, \gamma_{\tau_1^2}, \alpha, \rho, S^2) \Big) \Big) \end{aligned} \quad (10.3)$$

which holds below $\gamma_{\tau_1^2+1}$, since it holds for $\alpha = \alpha_{\tau_2}$ below α_1 . Hence, (10.2), (10.3) imply that S^2 is rejected at the defining of the matrix value $\alpha S_{\tau_1^2}$ by definition 8.3 2).

But it can happen only if S^2 on α is *suppressed* for $\gamma_{\tau_1^2}$, that implies $SIN_n(\gamma_{\tau_1^2})$ contrary to (10.1).

Hence

$$\exists \tau < \tau_2 \quad \forall \tau' \in [\tau, \tau_2[\quad a_{\tau'} = 1,$$

and from lemma 9.4 it follows $a_\tau \equiv 1$ on $] \tau_1, \tau_2[$.

—

From here and theorem 1 it comes directly

Corollary 10.2

Let

$$(i) \quad A_2^{\leq \alpha_1}(\tau_1, \tau_2, \tau_3);$$

$$(ii) \quad] \gamma_{\tau_1}^{< \alpha_1}, \gamma_{\tau_2}^{< \alpha_1}] \cap SIN_n^{< \alpha_1} \neq \emptyset.$$

Then

$$1) \quad \gamma_{\tau_2}^{< \alpha_1} \text{ is the successor of } \gamma_{\tau_1}^{< \alpha_1} \text{ in } SIN_n^{< \alpha_1};$$

$$2) \quad a_\tau^{< \alpha_1} \equiv 1 \text{ on }] \tau_1, \tau_2[, \quad a_{\tau_2}^{< \alpha_1} = 0 \quad \text{and}$$

$$3) \quad \tilde{\delta}_{\tau_2}^{< \alpha_1} \leq \gamma_{\tau_1}^{< \alpha_1}. \quad 14)$$

—

Lemma 10.3

Let

$$(i) \quad A_2^{\leq \alpha_1}(\tau_1, \tau_2, \tau_3);$$

$$(ii) \quad a_{\tau_2}^{< \alpha_1} = 1.$$

Then for the prejump cardinal $\alpha^2 = \alpha_{\tau_2}^{<\alpha_1 \Downarrow}$ there exists an ordinal

$$\tau_* = \min\{\tau \in]\tau_1, \tau_2[: \gamma_\tau^{<\alpha_1} \in SIN_n^{<\alpha^2}\}$$

such that

- 1) $\gamma_{\tau_*}^{<\alpha_1} < \tilde{\delta}_{\tau_2}^{<\alpha_1}$, $\gamma_{\tau_*}^{<\alpha_1} \notin SIN_n^{<\alpha_1}$;
- 2) $a_\tau^{<\alpha^2} \equiv a_\tau^{<\alpha_1} \equiv 1$ on $] \tau_1, \tau_*[$;
- 3) $\alpha S_f^{<\alpha^2}$ is monotone on $] \tau_1, \tau_*[$ and
- 4) $Od\alpha S_f^{<\alpha_1}(\tau_1, \tau_*) > Od(\alpha S_{\tau_2}^{<\alpha_1})$.

Proof. The upper indices $< \alpha_1, \triangleleft \alpha_1$ will be omitted. First one should see that $\gamma_{\tau_1} < \tilde{\delta}_{\tau_2}$, otherwise (ii) and lemma 9.2 break (i).

Then from lemma 10.1 and (i), (ii) it comes the existence of the following ordinal below the prejump cardinal $\alpha^2 = \alpha_{\tau_2}^\Downarrow$:

$$\tau_* = \min\{\tau > \tau_1 : \gamma_\tau \in (SIN_n^{<\alpha^2} - SIN_n)\}.$$

On $] \tau_1, \tau_*[$ αS_f is monotone and by theorem 1 (for α^2 instead of α_1) γ_{τ_*} is the successor of γ_{τ_1} in $SIN_n^{<\alpha^2}$. Due to lemma 9.3 the function αS_f stabilizes on $] \tau_1, \tau_*[$, so that for some $\tau_0 \in] \tau_1, \tau_*[$, S^0 the proposition $\forall \tau \geq \tau_0 \alpha S_\tau = S^0$ is true below γ_{τ_*} . Hence, below γ_{τ_*} the weaker proposition is true:

$$\forall \tau \left(\tau_0 < \tau \rightarrow \exists S (S = \alpha S_\tau \wedge S \geq S^0) \right).$$

It can be formulated in the Π_n -form, just as it was done in the proof of lemma 9.3, by means of proposition (9.4), where $\tau_{1,3}^n$, $\alpha S_{\tau_2}^1$ should be replaced with τ_0 , S^0 respectively.

The cardinal γ_{τ_*} extends this proposition up to α^2 and therefore by (i)

$$\rho = Od\alpha S_f(\tau_1, \tau_*) > Od(\alpha S_{\tau_2}).$$

Next, let us discuss the unit characteristic. From lemma 3.2 [27] and (i), (ii) it comes that there exist some admissible carriers of α -matrices of unit characteristic disposed cofinally to γ_{τ_1} as it was several times above. Hence such carriers must be also in $]\gamma_{\tau_*}, \alpha^2[$, otherwise γ_{τ_*} would be defined below α^2 along with the ordinal ρ and then by lemma 4.6 [27]

$$\rho < Od(\alpha S_{\tau_2}).$$

It remains to apply lemmas 3.2 [27], 9.4 (where τ_* , α^2 play the role of τ_2 , α_1 respectively), since $\alpha S_f^{<\alpha^1}$, $a_f^{<\alpha^1}$ coincide with $\alpha S_f^{<\alpha^2}$, $a_f^{<\alpha^2}$ on $[\tau_1, \tau_*[$ due to lemma 8.7 about absoluteness.

—

With the help of reasoning analogous to the proofs of lemmas 10.1-10.3 it is not hard to obtain

Lemma 10.4

Let

- (i) $A_2^{<\alpha_1}(\tau_1, \tau_2, \tau_3)$;
- (ii) $\forall \gamma < \gamma_{\tau_2}^{<\alpha_1} \exists \tau (\gamma < \gamma_{\tau}^{<\alpha^2} \wedge a_{\tau}^{<\alpha^2} = 1)$ for $\alpha^2 = \alpha_{\tau_2}^{<\alpha_1 \Downarrow}$;
- (iii) $a_{\tau_2}^{<\alpha_1} = 0$.

Then

- 1) $\tilde{\delta}_{\tau_2}^{<\alpha_1} \leq \gamma_{\tau_1}^{<\alpha_1}$ and
- 2) $\exists \tau \in]\tau_1, \tau_2[(a_{\tau}^{<\alpha_1} = 1 \wedge \alpha S_{\tau}^{<\alpha_1} \gg \alpha S_{\tau_2}^{<\alpha_1})$. ¹⁵⁾

Proof. The upper indices $< \alpha_1$, $\triangleleft \alpha_1$ will be omitted. Let us reveal the situation below, standing as usual on $\alpha^2 = \alpha_{\tau_2}^{<\alpha_1 \Downarrow}$. Suppose that $\tilde{\delta}_{\tau_2} \in]\gamma_{\tau_1}, \gamma_{\tau_2}[$; here the following two cases should

be considered:

1. $[\gamma_{\tau_1}, \gamma_{\tau_2}[\cap SIN_n^{<\alpha^2} \subseteq SIN_n$, then again (just as it was in the proof of theorem 1) αS_f is monotone on the interval $[\tau_1, \tau_2[$ such that the interval $]\gamma_{\tau_1}, \gamma_{\tau_2}[$ contains SIN_n -cardinals, contrary to theorem 1.
2. $[\gamma_{\tau_1}, \gamma_{\tau_2}[\cap SIN_n^{<\alpha^2} \not\subseteq SIN_n$. In this case one should again apply the restriction-and-extension techniques of reasoning precisely as it was done in part 1a. of lemma 9.4 proof. Let us first repeat the argument from the proof of lemma 10.3 concerning the function αS_f defined on the interval $]\tau_1, \tau_*[$, where

$$\gamma_{\tau_*} = \min(SIN_n^{<\alpha^2} - SIN_n) \quad \text{and} \quad a_\tau \equiv 1 \quad \text{on} \quad]\tau_1, \tau_*[$$

due to (ii). By theorem 1 (for τ_* , α^2 , as τ_2 , α_1) and (i) the cardinal γ_{τ_*} is the successor of γ_{τ_1} in $SIN_n^{<\alpha^2}$. Hence, the proposition

$$\forall \tau > \tau_1 \quad a_\tau = 1$$

is true below γ_{τ_*} ; it is not hard to see it with the help of condition (ii) and lemmas 3.2 [27], 9.4. This proposition can be formulated in the Π_n -form for this case, just as it was done above in the proof of lemma 9.4 in part 1a. with τ^3 as τ_1 :

$$\begin{aligned} \forall \gamma \Big(\gamma_{\tau_1} < \gamma \wedge SIN_{n-1}(\gamma) \longrightarrow \\ \longrightarrow \exists \delta, \alpha, \rho, S \left(SIN_n^{<\alpha^\flat}(\gamma_{\tau_1}) \wedge \alpha \mathbf{K}_{n+1}^\exists(1, \delta, \gamma, \alpha, \rho, S) \right) \Big). \end{aligned}$$

After that the $SIN_n^{<\alpha^2}$ -cardinal γ_{τ_*} extends this proposition up to α^2 and so $a_{\tau_2} = 1$ contrary to (iii).

Thus, $\tilde{\delta}_{\tau_2} \leq \gamma_{\tau_1}$; coming to the end of the proof one should apply lemma 9.2. Suppose that

$$\exists \tau \in]\tau_1, \tau_2[\quad \forall \tau' \in [\tau, \tau_2[\quad a_{\tau'} = 0,$$

then by this lemma the monotonicity of αS_f on $] \tau_1, \tau_2[$ implies that

$$Od\alpha S_f(\tau_1, \tau_2) \leq Od(\alpha S_{\tau_2})$$

contrary to (i). This contradiction along with (i) provides 2) and ends the proof.

—

The following lemma will be used at the end of the Main theorem proof, again relying strongly on the formula $A^{0 \triangleleft \alpha_1}(\tau)$ (recall definition 8.1 3.2):

$$\begin{aligned} \exists \gamma < \alpha_1 \Big(\gamma = \gamma_{\tau}^{< \alpha_1} \wedge \neg \exists a, \delta, \alpha, \rho < \alpha_1 \exists S \triangleleft \rho \big(\mathbf{K}_n^{\forall < \alpha_1}(\gamma, \alpha_{\chi}^{\Downarrow}) \wedge \\ \wedge \alpha \mathbf{K}_{n+1}^{\exists \triangleleft \alpha_1}(a, \delta, \gamma, \alpha, \rho, S) \big) \Big) \end{aligned}$$

meaning, that there is no α -matrix on some carrier $\alpha > \gamma_{\tau}^{< \alpha_1}$ admissible for $\gamma_{\tau}^{< \alpha_1}$ below α_1 .

Accordingly, through $A_1^{0 \triangleleft \alpha_1}(\tau_1, \tau_2, \alpha S_f^{< \alpha_1})$ is denoted the formula (remind definition 8.1 1.1 for $X_1 = \alpha S_f^{< \alpha_1}$):

$$A^{0 \triangleleft \alpha_1}(\tau_1) \wedge A_1^{\triangleleft \alpha_1}(\tau_1, \tau_2, \alpha S_f^{< \alpha_1}),$$

and also should be used the formula $A_2^{0 \triangleleft \alpha_1}(\tau_1, \tau'_2, \tau_3, \alpha S_f^{\triangleleft \alpha_1})$ (recall definition 8.2 3.3):

$$A^{0 \triangleleft \alpha_1}(\tau_1) \wedge A_2^{\triangleleft \alpha_1}(\tau_1, \tau'_2, \tau_3, \alpha S_f^{< \alpha_1}).$$

Lemma 10.5

Let

$$(i) \quad A_1^{0 \triangleleft \alpha_1}(\tau_1, \tau_2, \alpha S_f^{< \alpha_1});$$

(ii) $\tau_2 \leq \tau_3$ and S^3 be a matrix of characteristic a^3 on a carrier

$$\alpha_3 \in] \gamma_{\tau_3}^{< \alpha_1}, \alpha_1[$$

preserving $SIN_n^{<\alpha_1}$ -cardinals $\leq \gamma_{\tau_2}^{<\alpha_1}$ below α_1 and with generating eigendisseminator $\check{\delta}^{S^3}$;

$$(iii) \quad \check{\delta}^{S^3} \leq \gamma_{\tau_1}^{<\alpha_1}.$$

Then $a^3 = 0$.

Analogously for any disseminator $\check{\delta}$ of S^3 on α^3 with any base $\rho \geq \rho^{S^3}$.

Proof. As usual, we shall look over the situation below, standing on $a^3 = \alpha_3^\downarrow$ and considering the disseminator $\check{\delta}^{S^3}$ with the data base $\rho^3 = \rho^{S^3} = \hat{\rho}_1$, $\rho_1 = Od(S^3)$; the upper indices $< \alpha_1, \triangleleft \alpha_1$ will be dropped for some convenience.

Suppose that this lemma fails and $a^3 = 1$, then $\check{\delta}^{S^3}$ is admissible and nonsuppressed disseminator of S^3 on α_3 for any $\gamma_\tau \in]\tau_1, \tau_3[$ and by lemma 9.2 $a_\tau \equiv 1$ on $]\tau_1, \tau_2[$ and

$$Od\alpha S_f(\tau_1, \tau_2) \leq Od(\alpha S_{\tau_3}).$$

From lemma 9.5 it follows that due to (iii) αS_f is nonmonotone on $[\tau_1, \tau_2[$ (remind case 2. in the proof of point 3. of lemma 9.5) and that is why there exists τ'_2 for which there holds

$$A_2^0(\tau_1, \tau'_2, \tau_2), \quad a_{\tau'_2} = 1 \quad (10.4)$$

Now one should repeat precisely the reasoning from part 1b. of lemma 9.4 proof. Below α^3 the following Σ_{n+1} -proposition is true by lemma 8.7 about absoluteness (remind (9.8)):

$$\begin{aligned} \exists \gamma^0 \exists \tau'_1, \tau''_2, \tau'_3 < \gamma^0 \left(SIN_n(\gamma^0) \wedge \gamma_{\tau'_1}^{<\gamma^0} < \gamma_{\tau''_2}^{<\gamma^0} < \gamma_{\tau'_3}^{<\gamma^0} < \gamma^0 \wedge \right. \\ \left. \wedge A_2^{0 \triangleleft \gamma^0}(\tau'_1, \tau''_2, \tau'_3, \alpha S_f^{<\gamma^0}) \wedge \forall \tau''' \in]\tau'_1, \tau''_2] a_{\tau'''}^{<\gamma^0} = 1 \wedge \right. \\ \left. \wedge \alpha S_{\tau''_2}^{<\gamma^0} = \alpha S_{\tau'_2} \right). \end{aligned} \quad (10.5)$$

It contains the constants $< \rho^3$ and $\alpha S_{\tau'_2} \triangleleft \rho^3$, therefore the disseminator $\check{\delta}^{S^3}$ restricts this proposition and it fulfills below $\check{\delta}^{S^3}$.

Now let us reveal the situation below the prejump cardinal

$$\alpha^2 = \alpha_{\tau'_2}^{\downarrow}.$$

By (i) $\gamma_{\tau_1} \in SIN_n$, therefore by lemma 8.5 1) $\gamma_{\tau_1} \in SIN_n^{<\alpha^3}$. Since $\check{\delta}^{S^3} \leq \gamma_{\tau_1}$ and $\check{\delta}^{S^3} \in SIN_n^{<\alpha^3}$, lemma 3.8 [27] (for α^3 , γ_{τ_1} as α_1 , α_2) implies $\check{\delta}^{S^3} = \gamma_{\tau_1}$ or $\check{\delta}^{S^3} \in SIN_n^{<\gamma_{\tau_1}}$; then by the same lemma (for γ_{τ_1} as α_2) $\check{\delta}^{S^3} \in SIN_n$.

From here and lemma 8.5 1) it comes out $\check{\delta}^{S^3} \in SIN_n^{<\alpha^2}$; hence, in proposition (10.5) we can replace γ^0 with α^2 by lemma 8.7 about absoluteness and due to lemma 8.5 5) (for $\alpha_{\tau'_2}$ as α) it comes out $a_{\tau'_2} = 0$ contrary to (10.4).

—

Now the special theory of matrix functions is developed enough to start the proof of the Main theorem.

Chapter III

Applications of Special Theory

11 Proof of Main Theorem

The contradiction, which proves the Main theorem, is the following:

On one hand, by lemma 8.9 the function $\alpha S_f^{<\alpha_1}$ is defined on the nonempty set

$$T^{\alpha_1} = \{\tau : \alpha\delta^* < \gamma_\tau < \alpha_1\}$$

for every sufficiently great cardinal $\alpha_1 \in SIN_n$.

Its monotonicity on this set is excluded by theorem 1.

But on the other hand, this monotonicity is ensured by the following theorem for every SIN_n -cardinal $\alpha_1 > \alpha\delta^*$ of sufficiently great cofinality. Remind, that bounding cardinals α_1 are always assumed to be equiformative with χ^* , that is there holds $A_6^e(\chi^*, \alpha_1)$ (recall definition 8.1 5.1 for $\chi = \chi^*, \alpha^0 = \alpha_1$).

Theorem 2.

Let the function $\alpha S_f^{<\alpha_1}$ be defined on nonempty set

$$T^{\alpha_1} = \{\tau : \gamma_{\tau_1}^{<\alpha_1} < \gamma_\tau^{<\alpha_1} < \alpha_1\}$$

such that $\alpha_1 < k$ and:

$$(i) \quad \tau_1 = \min\{\tau : \forall \tau' (\gamma_\tau^{<\alpha_1} < \gamma_{\tau'}^{<\alpha_1} \longrightarrow \tau' \in \text{dom}(\alpha S_f^{<\alpha_1}))\};$$

$$(ii) \quad \sup SIN_n^{<\alpha_1} = \alpha_1;$$

(iii) $cf(\alpha_1) \geq \chi^{*+}$.

Then $\alpha S_f^{<\alpha_1}$ is monotone on this set:

$$\forall \tau_1, \tau_2 \in T^{\alpha_1} (\tau_1 < \tau_2 \rightarrow \alpha S_{\tau_1}^{<\alpha_1} \leq \alpha S_{\tau_2}^{<\alpha_1}).$$

Proof. The scheme of the proof is the following in outline.

The reasoning will be carried out by the induction on the cardinal α_1 .

Let us suppose, that this theorem fails and the cardinal α_1^* is *minimal* breaking this theorem, that is the function $\alpha S_f^{<\alpha_1^*}$ is nonmonotone on the set

$$T^{\alpha_1^*} = \{\tau : \gamma_{\tau_1^*}^{<\alpha_1^*} < \gamma_\tau^{<\alpha_1^*} < \alpha_1^*\}$$

with specified properties (i)–(iii) for some τ_1^* , so that the *first inductive hypothesis* holds:

for every $\alpha_1 < \alpha_1^*$ the function $\alpha S_f^{<\alpha_1}$ is monotone on the set T^{α_1} with properties (i)–(iii).

It follows straight from theorem 1, that this α_1^* is simply the *minimal* cardinal α_1 , for which the set T^{α_1} exist, because for every such $\alpha_1 < \alpha_1^*$ the function $\alpha S_f^{<\alpha_1}$ on T^{α_1} is *nonmonotone* by theorem 1 and at the same time is *monotone* by the minimality of α_1^* .

The reasoning will be conducted below α_1^* (and all variables will be bounded by α_1^*), or below bounding cardinals $\alpha_1 \leq \alpha_1^*$, so the upper indices $< \alpha_1^*$, $\triangleleft \alpha_1^*$ will be omitted for some shortness up to the end of theorem 2 proof.

First, let us notice that in conditions of this theorem 2 there holds

$$\gamma_{\tau_1}^{<\alpha_1} \in SIN_n^{<\alpha_1};$$

to see it one should repeat once more the argument applied before several times (first in the proofs of lemmas 7.7, 8.10). Therefore

it is not hard to see that for *every* sufficiently great $\tau_3^* \in T^{\alpha_1^*}$ the interval $[\gamma_{\tau_1^*}, \gamma_{\tau_3^*}[$ can be considered *as the block*, that is there exist some ordinals $\tau_1^{*'}, \tau_2^*, \eta^{*3}$ which fulfill the statement (remind definition 8.1 1.6 for $X_1 = \alpha S_f, X_2 = a_f$):

$$A_4^b(\tau_1^*, \tau_1^{*'}, \tau_2^*, \tau_3^*, \eta^{*3}, \alpha S_f, a_f).$$

Here (due to this definition 8.1) $\tau_1^{*'}$ is the index of the matrix $\alpha S_{\tau_1^{*'}}$ of *unit characteristic* $a_{\tau_1^{*'}} = 1$ on its carrier $\alpha_{\tau_1^{*'}}$ and η^{*3} — the type of this interval.

Next, due to condition (iii) of this theorem 2 we can use the index $\tau_3^* \in T^{\alpha_1^*}$ such, that the interval $[\gamma_{\tau_1^*}, \gamma_{\tau_3^*}[$ has just the type

$$\eta^{*3} > Od(\alpha S_{\tau_1^{*'}}), \quad \eta^{*3} < \chi^{*+}.$$

Now the formula \mathbf{K}^0 starts to work and *closes the diagonal reasoning*:

There arises the matrix $\alpha S_{\tau_3^*}$ on the carrier $\alpha_{\tau_3^*}$ along with its disseminator $\tilde{\delta}^{*3} = \tilde{\delta}_{\tau_3^*}^*$ and data base $\rho^{*3} = \rho_{\tau_3^*}^*$, and we shall see, that by lemma 10.5 it has *zero* characteristic on this carrier. Standing on the prejump cardinal $\alpha^{*3} = \alpha_{\tau_3^*}^{\downarrow}$ one should reveal the following situation below α^{*3} :

One shall see soon, that by lemma 8.8 disseminator $\tilde{\delta}^{*3}$ falls into some maximal block $[\gamma_{\tau_1^*}, \gamma_{\tau_3^*}[$ below α^{*3} of a type $\eta^{*3'} < \chi^{*+}$, where γ^{*3} is some $\gamma_{\tau_3^{*'}}^{<\alpha^{*3}}$. It is not hard to see, that

$$\gamma_{\tau_3^*}^* \leq \gamma^{*3} \quad \wedge \quad \eta^{*3} \leq \eta^{*3'};$$

so, there holds

$$A_4^{Mb < \alpha^{*3}}(\tau_1^*, \tau_1^{*'}, \tau_2^*, \tau_3^{*'}, \eta^{*3'}, \alpha S_f^{<\alpha^{*3}}, a_f^{<\alpha^{*3}}).$$

All these facts together constitute the premise of lemma 8.5 6):

$$a_{\tau_3^*} = 0 \wedge \gamma_{\tau_1^*}^{<\alpha^{*3}} \leq \tilde{\delta}^{*3} < \gamma_{\tau_3^{*'}}^{<\alpha^{*3}} \wedge \\ \wedge A_4^{Mb \triangleleft \alpha^{*3}}(\tau_1^*, \tau_1^{*'}, \tau_2^*, \tau_3^{*'}, \eta^{*3'}, \alpha S_f^{<\alpha^{*3}}, a_f^{<\alpha^{*3}}).$$

Therefore this lemma implies

$$\eta^{*3'} < \rho^{*3} \vee \rho^{*3} = \chi^{*+};$$

thus, at any rate,

$$Od(\alpha S_{\tau_1^{*'}}) < \eta^{*3} \leq \eta^{*3'} < \rho^{*3}.$$

But we shall see soon, that it is impossible, because by lemma 9.5 (about stairway cut-off from above) and lemma 11.3 below there holds:

$$\rho^{*3} \leq Od(\alpha S_{\tau_1^{*'}}).$$

This contradiction will end the proof of theorem 2.

To realize this scheme some more information is needed.

The reasoning sketched above relies on the following easy auxiliary lemmas 11.1, 11.3, which are coming as its carrying construction and are describing some important properties of zero matrices behavior; they were not presented earlier because of their rather special character. For this purpose one should remind the formula (recall definition 8.1 1.1 for $X_1 = \alpha S_f^{<\alpha_1}$)

$$A_1^{<\alpha_1}(\tau_1, \tau_2, \alpha S_f^{<\alpha_1}) :$$

$$\begin{aligned} \tau_1 + 1 < \tau_2 \wedge \tau_1 = \min\{\tau :]\tau, \tau_2[\subseteq \text{dom}(\alpha S_f^{<\alpha_1})\} \\ \wedge \gamma_{\tau_1}^{<\alpha_1} \in \text{SIN}_n^{<\alpha_1} \wedge \gamma_{\tau_2}^{<\alpha_1} \in \text{SIN}_n^{<\alpha_1}; \end{aligned}$$

Remind also that we often omit the functions $\alpha S_f^{<\alpha_1}, a_f^{<\alpha_1}$ in notations of formulas below α_1 ; recall that the type of interval $[\gamma_{\tau_1}^{<\alpha_1}, \gamma_{\tau_2}^{<\alpha_1}[$ below α_1 is the order type of the set (see definition 8.1 1.3):

$$\{\gamma : \gamma_{\tau_1}^{<\alpha_1} < \gamma < \gamma_{\tau_2}^{<\alpha_1} \wedge \text{SIN}_n^{<\alpha_1}(\gamma)\}.$$

Beforehand one should delay on the following auxiliary arguments, suitable to shorten the succeeding reasonings; with this aim one should introduce the following notion:

an interval $[\tau_1, \tau_2[$ and the corresponding interval $[\gamma_{\tau_1}^{<\alpha_1}, \gamma_{\tau_2}^{<\alpha_1}[$ will be called the intervals of matrix admissibility, or *admissibility intervals*, below α_1 , if for every $\tau' \in]\tau_1, \tau_2[$ there exist some α -matrix S on some carrier $> \gamma_{\tau'}^{<\alpha_1}$ *admissible* for $\gamma_{\tau'}^{<\alpha_1}$ below α_1 :

$$\forall \tau' \in]\tau_1, \tau_2[\exists a', \delta', \alpha', \rho', S' \alpha \mathbf{K}^{<\alpha_1}(a', \delta', \gamma_{\tau'}^{<\alpha_1}, \alpha', \rho', S'),$$

and $\gamma_{\tau_1}^{<\alpha_1} \in \text{SIN}_n, \gamma_{\tau_2}^{<\alpha_1} \in \text{SIN}_n$ and τ_1 is the minimal ordinal with these attributes.

Next, the following properties of any zero matrix S on its carrier α admissible for $\gamma_{\tau}^{<\alpha_1}$ along with its *minimal* disseminator $\tilde{\delta}$ with base ρ below α_1 should be treated for $\alpha_1 \leq \alpha_1^*$:

(1a.) if $\gamma_{\tau_1}^{<\alpha_1} < \gamma_{\tau_2}^{<\alpha_1} \leq \gamma_{\tau}^{<\alpha_1}$ and $\tilde{\delta}$ falls in the admissibility interval $[\gamma_{\tau_1}^{<\alpha_1}, \gamma_{\tau_2}^{<\alpha_1}[$, that is $\gamma_{\tau_1}^{<\alpha_1} \leq \tilde{\delta} < \gamma_{\tau_2}^{<\alpha_1}$, then $\gamma_{\tau_1}^{<\alpha_1} = \tilde{\delta}$;

(1b.) if there exist some zero matrix S^1 on some *another* carrier $\alpha^1 \neq \alpha$, admissible for the same $\gamma_{\tau}^{<\alpha_1}$ along with its minimal disseminator $\tilde{\delta}^1$ with base ρ^1 , then S on α is *nonsuppressed* for $\gamma_{\tau}^{<\alpha_1}$ along with $\tilde{\delta}, \rho$ below α_1 .

The testing of these properties will be conducted by the induction on triples (α_1, α, τ) ordered lexicographically as usual (with α_1

as the first component, α as the second and τ as the third). Suppose the triple $(\alpha_1^0, \alpha^0, \tau^0)$ is the minimal violating (1a.) or (1b.); so the *second inductive hypothesis* is accepted: for every lesser triple (α_1, α, τ) there hold (1a.) and (1b.).

We shall see, that it provides contradictions; the reasoning forthcoming will be conducted below α_1^0 , so the upper indices $< \alpha_1^0$, $\triangleleft \alpha_1^0$ will be dropped as usual (up to the special remark, if the context will not point out to another case clearly).

1. Let us begin with (1a.); suppose it is wrong, that is there exist some zero matrix S^0 on its carrier $\alpha^0 > \gamma_{\tau^0}$, its minimal disseminator $\tilde{\delta}^0$ with base ρ^0 , all admissible for γ_{τ^0} , and $\tilde{\delta}^0$ falls into the *admissibility interval* $[\gamma_{\tau_1^0}, \gamma_{\tau_2^0}[$, but

$$\gamma_{\tau_1^0} < \tilde{\delta}^0 < \gamma_{\tau_2^0} \leq \gamma_{\tau^0}, \quad \text{that is } \tilde{\delta}^0 = \gamma_{\tau_3^0}, \tau_1^0 < \tau_3^0 < \tau_2^0. \quad (11.1)$$

From here and lemma 3.8 [27] it follows immediately, that

$$\tilde{\delta}^0 \in SIN_n$$

since $\tilde{\delta}^0 < \gamma_{\tau_2^0}$, $\tilde{\delta}^0 \in SIN_n^{<\alpha^{0\Downarrow}}$, $\gamma_{\tau_2^0} \in SIN_n$.

By definition of admissibility interval there exist the matrix $\alpha S_{\tau_3^0}^0$ on its carrier $\alpha_{\tau_3^0}^0$ admissible for $\gamma_{\tau_3^0}$ along with its minimal disseminator $\tilde{\delta}_{\tau_3^0}^0$ with base $\rho_{\tau_3^0}^0$ (all below α_1^0).

From the second inductive hypothesis it comes $\tilde{\delta}_{\tau_3^0}^0 = \gamma_{\tau_1^0}$; hence lemma 3.2 [27] implies, that for every $\gamma_{\tau} \in]\gamma_{\tau_1^0}, \gamma_{\tau_3^0}[$ the matrix $\alpha S_{\tau_3^0}^0$ possesses many carriers $\alpha \in]\gamma_{\tau}, \gamma_{\tau+1}[$ admissible for γ_{τ} , which are nonsuppressed for this γ_{τ} due to the same inductive hypothesis and so there holds $A_1^0(\tau_1^0, \tau_3^0, \alpha S_f^{<\alpha_1^0})$:

$$A^0(\tau_1^0) \wedge A_1(\tau_1^0, \tau_3^0, \alpha S_f^{<\alpha_1^0})$$

below α_1^0 . The same arguments work below the prejump cardinal $\alpha^{0\Downarrow}$, so there holds below $\alpha^{0\Downarrow}$ as well:

$$A_1^{0<\alpha^{0\Downarrow}}(\tau_1^0, \tau_3^0, \alpha S_f^{<\alpha^{0\Downarrow}}).$$

It implies for the generating disseminator $\check{\delta}^0$ of S^0 on α^0 with the base ρ^0 :

$$\check{\delta}^0 \leq \gamma_{\tau_1^0}, \quad (11.2)$$

because in the opposite case $\check{\delta}^0$ falls strictly in the admissibility interval $]\gamma_{\tau_1^0}, \gamma_{\tau_3^0}[$:

$$\gamma_{\tau_1^0} < \check{\delta}^0 \leq \gamma_{\tau_3^0}, \quad (11.3)$$

and then $\check{\delta}^0$ extends up to $\alpha^{0\Downarrow}$ the Π_{n+1} -proposition about admissibility of some matrices for every $\gamma_{\tau}^{<\alpha^{0\Downarrow}} > \gamma_{\tau_1^0}$, that become even nonsuppressed for all such $\gamma_{\tau}^{<\alpha^{0\Downarrow}}$ by the second inductive hypothesis (all it below $\alpha^{0\Downarrow}$); hence, there arises the set $T^{\alpha^{0\Downarrow}}$ with properties (i)–(iii) specified in theorem 2, contrary to the first inductive hypothesis and theorem 1, that is to the minimality of α_1^* .

From this place the reasoning passes to the matter below $\alpha^{0\Downarrow}$, and the upper indices $< \alpha^{0\Downarrow}, \triangleleft \alpha^{0\Downarrow}$ will be dropped.

Below $\alpha^{0\Downarrow}$ the function αS_f is defined on the interval $]\tau_3^0, \tau_3^1[$ for $\gamma_{\tau_3^1}$ the successor of $\check{\delta}^0$ in SIN_n , by lemma 8.7 about absoluteness. From here and (11.2) it follows

$$\tau_3^0 \notin \text{dom}(\alpha S_f), \quad (11.4)$$

otherwise it again comes (11.3) or $\check{\delta}^0 = \gamma_{\tau_1^0}$ as the result of the minimizing of $\check{\delta}^0$ within $[\gamma_{\tau_1^0}, \gamma_{\tau_3^0}[$ contrary to the supposition (all it below $\alpha^{0\Downarrow}$).

But (11.4) can be carried out only when the admissible matrix $\alpha S_{\tau_3^0}$ is *suppressed* for $\gamma_{\tau_3^0}$, that is when there holds the suppression condition $A_5^{S,0}$ for $\alpha S_{\tau_3^0}$ on $\alpha_{\tau_3^0}$ of the characteristic $a_{\tau_3^0}$ with the base $\rho_{\tau_3^0}$ (see definition 8.1 2.6) below $\alpha^{0\Downarrow}$, — and now all boundings should be pointed out clearly:

$$a_{\tau_3^0} = 0 \wedge SIN_n^{<\alpha^{0\Downarrow}}(\gamma_{\tau_3^0}) \wedge \rho_{\tau_3^0} < \chi^{*+} \wedge \sigma(\chi^*, \alpha_{\tau_3^0}, S_{\tau_3^0}) \wedge$$

$$\begin{aligned}
& \wedge \exists \eta^* < \gamma_{\tau_3^0} \left(A_{5.4}^{sc \triangleleft \alpha^{0\downarrow}}(\gamma_{\tau_3^0}, \eta^*, \alpha S_f^{< \alpha^{0\downarrow}} | \tau_3^0, a_f^{< \alpha^{0\downarrow}} | \tau_3^0) \wedge \right. \\
& \quad \wedge \forall \tau' (\gamma_{\tau_3^0} < \gamma_{\tau'}^{< \alpha^{0\downarrow}} \wedge SIN_n^{< \alpha^{0\downarrow}}(\gamma_{\tau'}^{< \alpha^{0\downarrow}}) \rightarrow \\
& \rightarrow \exists \alpha', S' [\gamma_{\tau'}^{< \alpha^{0\downarrow}} < \alpha' < \gamma_{\tau'+1}^{< \alpha^{0\downarrow}} \wedge SIN_n^{< \alpha'^{\downarrow}}(\gamma_{\tau'}^{< \alpha^{0\downarrow}}) \wedge \sigma(\chi^*, \alpha', S') \wedge \\
& \quad \left. \wedge A_{5.5}^{sc \triangleleft \alpha^{0\downarrow}}(\gamma_{\tau_3^0}, \eta^*, \alpha'^{\downarrow}, \alpha S_f^{< \alpha'^{\downarrow}}, a_f^{< \alpha'^{\downarrow}} | \tau_3^0) \right] \Big).
\end{aligned} \tag{11.5}$$

Hence, there exist the cardinals

$$\gamma^m < \gamma^* \leq \gamma_{\tau_1^0} < \gamma_{\tau_3^0} \quad \text{and the limit type } \eta^*$$

which carry out all its constituents $A_{5.1}^{sc} - A_{5.5}^{sc}$ below $\alpha^{0\downarrow}$ (see definition 8.1 2.1–2.5); in particular the interval $[\gamma_{\tau_1^0}, \gamma_{\tau_3^0}[$ is the block of the type η^* due to the condition

$$A_4^{b < \alpha^{0\downarrow}}(\tau_1^0, \tau_3^0, \eta^*, \alpha S_f^{< \alpha^{0\downarrow}} | \tau_3^0, a_f^{< \alpha^{0\downarrow}} | \tau_3^0)$$

from the condition $A_{5.4}^{sc}$ (see definition 8.1 2.4, 2.3). Moreover, there exist some its succeeding maximal block

$$[\gamma_{\tau_3^0}, \gamma_{\tau_2^0}^{< \alpha^{0\downarrow}}[\text{ of the type } \geq \eta^* \text{ below } \alpha^{0\downarrow}. \tag{11.6}$$

Really, let us take any cardinal

$$\gamma' = \gamma_{\tau_2'}^{< \alpha^{0\downarrow}} \in SIN_n^{< \alpha^{0\downarrow}}, \tau_2' > \tau_2$$

such that below $\alpha^{0\downarrow}$

$$\tau_2' \notin \text{dom}(\alpha S_f^{< \alpha^{0\downarrow}}). \tag{11.7}$$

Then by (11.5) there exist some singular matrix S' on its indicated carrier $\alpha' > \gamma'$ with prejump cardinal α'^{\downarrow} preserving all $SIN_n^{< \alpha^{0\downarrow}}$ -cardinals $\leq \gamma'$ and carrying out the condition below α'^{\downarrow} (recall definition 8.1 2.5):

$$A_{5.5}^{sc}(\gamma_{\tau_3^0}, \eta^*, \alpha'^{\downarrow}, \alpha S_f^{< \alpha'^{\downarrow}}, a_f^{< \alpha'^{\downarrow}});$$

it means, that the whole interval $[\gamma_{\tau_3^0}, \alpha'^{\Downarrow}]$ is covered by blocks below α'^{\Downarrow} of types $\geq \eta^*$. Among them there exist the *succeeding* block

$$[\gamma_{\tau_3^0}, \gamma_{\tau_2^3}^{<\alpha'^{\Downarrow}}] \text{ of the type } \geq \eta^*,$$

so one can treat its subblock $[\gamma_{\tau_3^0}, \gamma_{\tau_2^4}^{<\alpha'^{\Downarrow}}]$ of the type exactly η^* . Remind, the type η^* is limit, thereafter for every γ_τ from this subblock there exist many different matrix carriers admissible for such γ_τ due to lemma 3.2 [27] about restriction; after that due to the second inductive hypothesis all of them are nonsuppressed for all such corresponding γ_τ – and all it below α'^{\Downarrow} .

The same argument works below $\alpha^{0\Downarrow}$ and we return to the situation below this cardinal. From (11.7) it follows

$$\gamma_{\tau_2^4}^{<\alpha'^{\Downarrow}} = \gamma_{\tau_2^4}^{<\alpha^{0\Downarrow}}, \quad \gamma_{\tau_2^4}^{<\alpha^{0\Downarrow}} \in SIN_n^{<\alpha^{0\Downarrow}}$$

and the interval $[\gamma_{\tau_3^0}, \gamma_{\tau_2^4}^{<\alpha^{0\Downarrow}}]$ is really the block of the type η^* , but already below $\alpha^{0\Downarrow}$, which contains the admissible disseminator $\tilde{\delta}^0$ of the matrix S^0 on α^0 .

But it provides the contradiction. On one hand, S^0 is admissible for γ_{τ^0} and then by the closing condition \mathbf{K}^0 it has the disseminator $\tilde{\delta}^0$ with base $\rho^0 > \eta^*$. But on the other hand, the preceding block $[\gamma_{\tau_1^0}, \gamma_{\tau_3^0}]$ below $\alpha^{0\Downarrow}$ has the same type η^* and by (11.2) its left end $\gamma_{\tau_1^0}$ can serve as the admissible disseminator for S^0 on α^0 with the same base ρ^0 , and thereby $\tilde{\delta}^0 \leq \gamma_{\tau_1^0}$ due to the minimality of $\tilde{\delta}^0$, contrary to supposition (11.1).

2. So, (1a.) holds for $(\alpha_1^0, \alpha^0, \tau^0)$ and it remains to suppose that (1b.) is wrong for this triple, and we return to the matter below α_1^0 ; it means:

there exist some zero matrix S^{01} on the carrier $\alpha^{01} \neq \alpha^0$ admissible for $\gamma^0 = \gamma_{\tau^0}$ along with its minimal disseminator $\tilde{\delta}^{01}$ and the generating disseminator $\check{\delta}^{01}$ with the base ρ^{01} ,

but still S^0 on α^0 is suppressed for $\gamma^0 = \gamma_{\tau^0}$ (below α_1^0); we shall consider the *minimal* α^{01} with this property.

Since zero matrix S^0 on α^0 is admissible for γ^0 , this suppression means, that there holds the suppression condition (11.5) below α_1^0 , that is for $\alpha^{0\downarrow}$, $\gamma_{\tau_3^0}$ replaced with α_1^0 , γ^0 respectively everywhere in (11.5).

From here it follows that

$$\alpha^0 < \alpha^{01},$$

because if $\alpha^0 > \alpha^{01}$, then the second inductive hypothesis states, that S^{01} on α^{01} is nonsuppressed for γ^0 below α_1^0 , and at the same time it is suppressed by the same suppression condition. Besides that S^0 on α^0 is the only matrix admissible for γ^0 with the carrier $\alpha^0 \in]\gamma^0, \alpha^{01}[$ due to the minimality of α^{01} .

Now this condition (11.5) with α_1^0, γ^0 instead of $\alpha^{0\downarrow}, \gamma_{\tau_3^0}$ respectively states the existence of the cardinals (we preserve the previous notations to stress the analogy with the reasoning in part 1.):

$$\gamma^m < \gamma^* \leq \gamma_{\tau_1^0} < \gamma^0 \text{ and the limit type } \eta^*,$$

holding all the constituents $A_{5,1}^{sc} - A_{5,5}^{sc}$; in particular the interval $[\gamma^m, \gamma^*[$ is covered by the maximal blocks of types nondecreasing substantially up to the limit ordinal η^* ; $[\gamma^*, \gamma_{\tau_1^0}[$ is covered by the maximal blocks of the type exactly η^* ; $[\gamma_{\tau_1^0}, \gamma^0[$ is also the block of the same type η^* – and so on.

All these conditions define $\gamma^m, \gamma^*, \gamma_{\tau_1^0}, \eta^*$ uniquely through γ^0 below α_1^0 and provide the very special kind of this covering; to operate with it one should use the following auxiliary Σ_n -formulas treating only the notion of admissibility (remind definition 8.2 5):

$$\alpha\mathbf{K}^1(\gamma) : \exists \alpha', S' \alpha\mathbf{K}(\gamma, \alpha', S');$$

$$\alpha\mathbf{K}^2(\gamma) : \exists \alpha', S' \exists \alpha'', S'' (\alpha' \neq \alpha'' \wedge \wedge \alpha\mathbf{K}(\gamma, \alpha', S') \wedge \alpha\mathbf{K}(\gamma, \alpha'', S'')).$$

The first of them means, that there exist at least *one* matrix carrier α' admissible for γ ; the second – that there exist *more than one* such carriers $\alpha' \neq \alpha''$; thus $\neg \alpha \mathbf{K}^1(\gamma)$ means, that there is no such carriers at all.

Since the type η^* is limit, every maximal block $[\gamma_{\tau_1}, \gamma_{\tau_2}[$ from the covering of $[\gamma^*, \gamma_{\tau_1^0}[$ possesses two properties:

(i) if γ_τ is inner in $[\gamma_{\tau_1}, \gamma_{\tau_2}[$, $\tau_1 < \tau < \tau_2$, then $\alpha \mathbf{K}^2(\gamma_\tau)$ holds; it follows from the second inductive hypothesis and lemma 3.2 [27] about restriction;

(ii) if γ_τ is the end of this block, then $\alpha \mathbf{K}^1(\gamma_\tau)$ fails.

It can be verified in the following way. Suppose that γ_τ is the right end, $\gamma_\tau = \gamma_{\tau_2}$, then the existence of some S' on α' admissible for γ_{τ_2} provides the union of this block and of the succeeding block $[\gamma_{\tau_2}, \gamma_{\tau_3}[$ in the common admissibility interval $[\gamma_{\tau_1}, \gamma_{\tau_3}[$ of the type $2\eta^*$. And again by the second inductive hypothesis and lemma 3.2 [27] there exist several matrix carriers α' admissible for γ_{τ_2} which become nonsuppressed for γ_{τ_2} and, so, the function αS_f is defined on the whole interval $[\tau_1, \tau_3[$, though $[\gamma_{\tau_1}, \gamma_{\tau_2}[$ is the *maximal* block (all it below α_1^0). The left end $\gamma_\tau = \gamma_{\tau_1}$ should be treated in the analogous way.

Hence for every $\gamma_\tau \in [\gamma^*, \gamma^0[$ there holds the Δ_{n+1} -formula:

$$\alpha \mathbf{K}^2(\gamma_\tau) \vee \neg \alpha \mathbf{K}^1(\gamma_\tau); \quad (11.8)$$

it is not hard to see, that the same situation holds below $\alpha^{01\Downarrow}$ by the same reasons.

Now the generating disseminator $\check{\delta}^{01}$ of S^{01} on α^{01} starts to work and realizes the restriction-and-extension method.

First,

$$\tilde{\delta}^{01} \leq \gamma_{\tau_1^0};$$

in the opposite case

$$\gamma_{\tau_1^0} < \check{\delta}^{01} = \tilde{\delta}^{01} < \gamma^0$$

and $\check{\delta}^{01}$ extends up to $\alpha^{01\downarrow}$ the Σ_{n+1} -proposition

$$\forall \gamma_\tau > \gamma_{\tau_1^0} \quad \alpha \mathbf{K}^2(\gamma_\tau).$$

This fact along with the second inductive hypothesis provides the definiteness of $\alpha S_f^{<\alpha^{01\downarrow}}$ on some nonempty set $T^{\alpha^{01\downarrow}}$ with properties (i)–(iii) specified in theorem 2, contrary to the minimality of α_1^* .

From $\check{\delta}^{01} \leq \gamma_{\tau_1^0}$ it follows

$$\gamma^* < \check{\delta}^{01} \leq \tilde{\delta}^{01} \leq \gamma_{\tau_1^0}. \quad (11.9)$$

Really, the block $[\gamma_{\tau_1^0}, \gamma^0[$ obviously provides the following Σ_{n+1} -formula $\varphi(\tau_1^0, \tau^0, \eta^*)$ below α^{01} :

$$\exists \gamma (\gamma_{\tau_1^0} < \gamma_{\tau^0} \leq \gamma \wedge SIN_n(\gamma) \wedge A_{1.2}^{<\gamma}(\tau_1^0, \tau^0, \eta^*) \wedge$$

$$\wedge \forall \tau \in]\tau_1, \tau^0[\quad \alpha \mathbf{K}^{2<\gamma}(\gamma_\tau));$$

remind, here $A_{1.2}(\tau_1, \tau^0, \eta^*)$ means, that the interval $[\gamma_{\tau_1^0}, \gamma_{\tau^0}[$ has the type η^* .

The disseminator $\tilde{\delta}^{01}$ falls into $[\gamma_{\tau_1^0}, \gamma^0[$ and hence $\tilde{\delta}^{01} = \gamma_{\tau_1^0}$, otherwise $\tilde{\delta}^{01} < \gamma_{\tau_1^0}$ and by lemma 3.2 [27] there appear many carriers of S^{01} admissible for this $\gamma_{\tau_1^0}$; then by the second inductive hypothesis all of them are nonsuppressed for $\gamma_{\tau_1^0}$; hence the matrix function αS_f becomes defined for τ_1^0 contrary to the minimality of the left end $\gamma_{\tau_1^0}$ by definition of the block notion.

Since $\tilde{\delta}^{01} = \gamma_{\tau_1^0}$, the closing condition \mathbf{K}^0 for S^{01} on α^{01} implies $\eta^* < \rho^{01}$ for the base ρ^{01} of $\tilde{\delta}^{01}$.

But then the generating disseminator $\check{\delta}^{01}$ with this base restricts the Σ_{n+1} -proposition

$$\exists \tau'_1, \tau' \varphi(\tau'_1, \tau', \eta^*),$$

because it contains only constants, bounded by ρ^{01} .

Therefore below $\check{\delta}^{01}$ there appear blocks of types $\geq \eta^*$ (again due to second inductive hypothesis).

Now if $\check{\delta}^{01} \leq \gamma^*$, then it violates the condition $A_{5.2}^{sc}$ about non-decreasing of covering types of $[\gamma^m, \gamma^*[$ up to η^* substantially.

Thus (11.9) holds. Due to (11.8) below $\check{\delta}^{01}$ there holds the Π_{n+1} -proposition

$$\forall \tau (\gamma^* < \gamma_\tau \rightarrow (\alpha \mathbf{K}^2(\gamma_\tau) \vee \neg \alpha \mathbf{K}^1(\gamma_\tau)))$$

and the disseminator $\check{\delta}^{01}$ extends it up to $\alpha^{01\Downarrow}$ by lemma 6.6 [27] (for $m = n+1$, $\delta = \check{\delta}^{01}$, $\alpha_0 = \gamma^*$, $\alpha_1 = \alpha^{01\Downarrow}$). But it provides the contradiction: (11.8) holds for $\gamma_\tau = \gamma^0$, though there is exactly one matrix S^0 on α^0 below $\alpha^{01\Downarrow}$ admissible for γ^0 .

So, properties (1a.) and (1b.) are carried out. Now one can return to lemmas 8.5 8), 8.8, 8.10 (for $\alpha_1 \leq \alpha_1^*$):

(2) First one should dwell on lemma 8.8 1); now it is not hard to receive (8.5). Toward this end let us compare two intervals

$$]\gamma_{\tau_1}, \gamma_{\tau_2}[, \quad]\gamma_{\tau'_1}, \gamma_{\tau_2}[.$$

Due to (8.3), (8.4) $\tilde{\delta}^3$ contains in both of them, that provides

$$\gamma_{\tau'_1} \leq \gamma_{\tau_1},$$

otherwise $\gamma_{\tau_1} < \gamma_{\tau'_1}$ and (8.3) causes the existence of some matrix S^1 on its carrier α^1 with the disseminator $\tilde{\delta}^1$ and base ρ^1 admissible for $\gamma_{\tau'_1}$. By lemma 3.2 [27] S^1 receives its carriers admissible for every $\gamma_\tau \in]\tilde{\delta}^1, \gamma_{\tau'_1}[$ along with the same $\tilde{\delta}^1, \rho^1$. So, there arises some admissibility interval $[\gamma_{\tau''_1}, \gamma_{\tau_3}[$ with $\gamma_{\tau''_1} < \gamma_{\tau'_1}$ and by (1a.) $\tilde{\delta}^3 = \gamma_{\tau''_1}$ contrary to (8.4). Thus there holds $\gamma_{\tau'_1} \leq \gamma_{\tau_1}$ and along with (8.3), (8.4) it implies (8.5), that provides the rest part of lemma 8.8 1) proof.

Turning to lemma 8.8 2), let α -matrix S of characteristic a on

a carrier α be admissible for $\gamma_\tau^{<\alpha_1}$ along with its disseminator $\tilde{\delta}$ and base ρ below α_1 ; one can prove

$$\{\tau' : \tilde{\delta} < \gamma_{\tau'}^{<\alpha_1} < \gamma_\tau^{<\alpha_1}\} \subseteq \text{dom}(\alpha S_f^{<\alpha_1})$$

by the reasoning already used above:

for every $\gamma_{\tau'}^{<\alpha_1} \in]\tilde{\delta}, \gamma_\tau^{<\alpha_1}[$ there exist many admissible carriers of the matrix S by lemma 3.2 [27], therefore all of them are nonsuppressed due to (1b.), hence $\tau' \in \text{dom}(\alpha S_f^{<\alpha_1})$.

By the similar reasons in lemma 8.10 the function $\alpha S_f^{<\alpha_1}$ is defined on the whole interval $] \alpha \tau_1^*, \alpha \tau^{*1}[$ for any $\alpha_1 > \alpha \delta^{*1}$, $\alpha_1 \in \text{SIN}_n$, and $\alpha \delta^* = \gamma_{\alpha \tau_1^*}$ is the disseminator of the matrix $\alpha S_{\alpha \tau^{*1}}^{<\alpha_1}$ on its carrier $\alpha_{\alpha \tau^{*1}}^{<\alpha_1}$ with the base $\alpha \rho^{*1} = \rho_{\alpha \tau^{*1}}^{<\alpha_1}$.

The similar reasoning in lemma 8.5 8) proof should be used. To finish this proof for the nonsuppressibility one should notice, that if S along with δ, ρ has a carrier α admissible and nonsuppressed for $\gamma_\tau^{<\alpha_1}$ only in $[\gamma_{\tau+1}^{<\alpha_1}, \alpha_1[$, then $\gamma_{\tau+1}^{<\alpha_1}$ restricts the $\text{SIN}_{n-1}^{<\alpha_1}$ -proposition

$$\exists \alpha' (\gamma < \alpha' \wedge \alpha \mathbf{K}_{n-2}(\delta, \gamma_\tau^n, \gamma_\tau^{<\alpha_1}, \alpha', \rho, S))$$

for any $\gamma \in]\gamma_\tau^{<\alpha_1}, \gamma_{\tau+1}^{<\alpha_1}[$, therefore S receives many carriers in $] \gamma_\tau^{<\alpha_1}, \gamma_{\tau+1}^{<\alpha_1}[$ admissible for $\gamma_\tau^{<\alpha_1}$, that also become nonsuppressed for $\gamma_\tau^{<\alpha_1}$ below α_1 by (1b.).

Next lemma shows that intervals $[\gamma_{\tau_1}^{<\alpha_1}, \gamma_{\tau_3}^{<\alpha_1}[$ of the matrix function $\alpha S_f^{<\alpha_1}$ definiteness with the minimal left end $\gamma_{\tau_1}^{<\alpha_1} \in \text{SIN}_n^{<\alpha_1}$ are composed in a special way: for every $\text{SIN}_n^{<\alpha_1}$ -cardinal $\gamma_\tau^{<\alpha_1} \in]\gamma_{\tau_1}^{<\alpha_1}, \gamma_{\tau_3}^{<\alpha_1}[$ the matrix $\alpha S_\tau^{<\alpha_1}$ has zero characteristic and disseminators $\check{\delta}_\tau \leq \tilde{\delta}_\tau = \gamma_{\tau_1}^{<\alpha_1}$ below α_1 :

Lemma 11.1

Let

$$(i) \ A_1^{<\alpha_1}(\tau_1, \tau_2);$$

(ii) S^2 be α -matrix of characteristic α^2 on a carrier

$$\alpha^2 \in]\gamma_{\tau_2}^{<\alpha_1}, \alpha_1[$$

admissible for $\gamma_{\tau_2}^{<\alpha_1}$ below α_1 along with its minimal disseminator $\tilde{\delta}^2$ with a base ρ^2 and with generating eigendisseminator $\check{\delta}^{S^2}$;

Then

$$\check{\delta}^{S^2} \leq \tilde{\delta}^2 = \gamma_{\tau_1}^{<\alpha_1} \quad \text{and} \quad \alpha^3 = 0.$$

Proof. The upper indices $< \alpha_1, \triangleleft \alpha_1$ will be dropped for shortness as usual.

Let us consider the α -matrix S^2 on its carrier α^2 admissible for γ_{τ_2} along with the minimal disseminator $\tilde{\delta}^2$ of the base ρ^2 and with the generating eigendisseminator $\check{\delta}^2 = \check{\delta}^{S^2}$, and examine the situation below the prejump cardinal $\alpha^{2\Downarrow}$.

1. Suppose that, on the contrary, this lemma is wrong and $\check{\delta}^2 \not\leq \gamma_{\tau_1}$, so it comes

$$\gamma_{\tau_1} < \check{\delta}^2 < \gamma_{\tau_2}.$$

By definition $\gamma_{\tau_2} \in SIN_n$ and, hence, $\gamma_{\tau_2} \in SIN_n^{<\alpha^{2\Downarrow}}$. Due to this fact and lemma 8.7 the admissibility below α_1 is equivalent to the admissibility below $\alpha^{2\Downarrow}$ for every $\gamma_\tau \in]\chi^*, \gamma_{\tau_2}[$.

Then the generating disseminator $\check{\delta}^2$ extends up to $\alpha^{2\Downarrow}$ the Π_{n+1} -proposition stating the definiteness of the function $\alpha S_f^{<\alpha^{2\Downarrow}}$, as it was several times above, for instance, in the form:

$$\forall \gamma' \left(\gamma_{\tau_1} < \gamma' \wedge SIN_{n-1}(\gamma') \rightarrow \exists \alpha, S \, \alpha \mathbf{K}(\gamma', \alpha, S) \right).$$

After that there appears the function $\alpha S_f^{<\alpha^{2\Downarrow}}$ defined on the set with properties (i)–(iii) from theorem 2:

$$T^{\alpha^{2\Downarrow}} = \{ \tau : \gamma_{\tau_1} < \gamma_\tau^{<\alpha^{2\Downarrow}} < \alpha^{2\Downarrow} \},$$

because for every $\tau \in T^{\alpha^{2\downarrow}}$ there appear some matrices on many carriers admissible for $\gamma_\tau^{<\alpha^{2\downarrow}}$ which are nonsuppressed for $\gamma_\tau^{<\alpha^{2\downarrow}}$ due to arguments (1b.), (1a.); but it contradicts the minimality of α_1^* .

2. So, $\tilde{\delta}^2 \leq \gamma_{\tau_1}$; moreover, there holds $A^0(\tau_1)$. Suppose it fails and there exist some α -matrix S^1 on its carrier α^1 admissible for γ_{τ_1} along with its minimal disseminator $\tilde{\delta}^1 = \gamma_{\tau_1^1}$ with base ρ^1 ; hence $\tau_1^1 < \tau_1$. By lemma 8.8 $]\tau_1^1, \tau_1[\subseteq \text{dom}(\alpha S_f)$ and, so, for every $\tau \in]\tau_1^1, \tau_2[$ there exist some α -matrix on a carrier admissible for γ_τ . Therefore there arises some admissibility interval $]\tau_1^{1'}, \tau_2[$ with $\tau_1^{1'} \leq \tau_1^1 < \tau_2$. Due to (1a.) $\tilde{\delta}^2 = \gamma_{\tau_1^{1'}}$ and by lemma 8.8 $]\tau_1^{1'}, \tau_2[\subseteq \text{dom}(\alpha S_f)$ contrary to the minimality of τ_1 stated here in (i).

3. Thus $A^0(\tau_1)$ and, hence, $A_1^0(\tau_1, \tau_2)$ hold on; thereby lemma 10.5 implies $a^2 = 0$. At last again by (1a.) there comes $\tilde{\delta}^2 = \gamma_{\tau_1}$.

—

Let us turn now to the following suitable notion that already was used above several times, but further it will play the key role; thereby it should be emphasized in the following

Definition 11.2

Let S be a matrix on some carrier α along with its disseminator $\tilde{\delta} < \gamma_\tau^{<\alpha_1}$ with a base ρ .

1) We say that S leans on $\tilde{\delta}$ on this carrier α below α_1 if $\tilde{\delta}$ falls in some block $[\gamma_{\tau_1}^{<\alpha_1}, \gamma_{\tau_3}^{<\alpha^\downarrow}]$ of type η , that is if there exist ordinals $\tau_1, \tau_1', \tau_2, \tau_3, \eta$ such that

$$\gamma_{\tau_1}^{<\alpha_1} \leq \tilde{\delta} < \gamma_{\tau_3}^{<\alpha^\downarrow} \wedge A_4^{b<\alpha^\downarrow}(\tau_1, \tau_1', \tau_2, \tau_3, \eta).$$

If in addition this block is the maximal below α^\downarrow and $\eta < \rho$ then

we say that S leans on $\tilde{\delta}$ very strongly.

2) Let a cardinal $v \in]\chi^*, \alpha^\Downarrow]$ be covered by blocks; this covering is called η -bounded (below α^\Downarrow) if all types of blocks in $]\chi^*, \alpha^\Downarrow]$ are bounded by some constant ordinal $\eta < \chi^{*+}$:

$$\begin{aligned} \forall \gamma' < v \ \forall \tau'_1, \tau'_2, \eta' (\chi^* < \gamma_{\tau'_1}^{<\alpha^\Downarrow} < \gamma_{\tau'_2}^{<\alpha^\Downarrow} < v \wedge \\ \wedge A_4^{Mb \triangleleft \alpha^\Downarrow}(\tau'_1, \tau'_2, \eta') \rightarrow \eta' \leq \eta). \end{aligned}$$

—

Part (I) of the following lemma comes out as the carrying construction of the further reasoning; part (II) will be used at the end-point of the proof of theorem 2 strongly. Here one should remind the notion of stairway and its various attributes, that were introduced just before lemma 9.5 by means of formulas 1.–8.; such stairway, being defined below the prejump cardinal $\alpha_1 = \alpha^\Downarrow$ of matrix S carrier α by means of the formula $A_8^{St \triangleleft \alpha^\Downarrow}(St, \alpha S_f^{<\alpha^\Downarrow}, a_f^{<\alpha^\Downarrow})$, should be used as the function on χ^{*+} :

$$St = ((\tau_1^\beta, \tau_*^\beta, \tau_2^\beta))_{\beta < \chi^{*+}},$$

so, that for every β, β_1, β_2 :

$$(i) \ \beta < \chi^{*+} \rightarrow \tau_1^\beta < \tau_*^\beta \leq \tau_2^\beta \wedge A_{1.1}^{Mst \triangleleft \alpha^\Downarrow}(\tau_1^\beta, \tau_*^\beta, \tau_2^\beta, \alpha S_f^{<\alpha^\Downarrow}, a_f^{<\alpha^\Downarrow}),$$

that is $[\gamma_{\tau_1^\beta}^{<\alpha^\Downarrow}, \gamma_{\tau_2^\beta}^{<\alpha^\Downarrow}[$ is the maximal unit step below α^\Downarrow :

$$A_{1.1}^{st \triangleleft \alpha^\Downarrow}(\tau_1^\beta, \tau_*^\beta, \tau_2^\beta, \alpha S_f^{<\alpha^\Downarrow}, a_f^{<\alpha^\Downarrow}) \wedge A_{1.1}^{M \triangleleft \alpha^\Downarrow}(\tau_1^\beta, \tau_2^\beta, \alpha S_f^{<\alpha^\Downarrow});$$

$$(ii) \ \beta_1 < \beta_2 < \chi^{*+} \longrightarrow$$

$$\longrightarrow \tau_2^{\beta_1} < \tau_1^{\beta_2} \wedge Od \alpha S_f^{<\alpha^\Downarrow}(\tau_1^{\beta_1}, \tau_*^{\beta_1}) < Od \alpha S_f^{<\alpha^\Downarrow}(\tau_1^{\beta_2}, \tau_*^{\beta_2}),$$

that is such steps are disposed successively one after another and their heights are strictly increasing;

$$(iii) \sup_{\beta} Od\alpha S_f^{<\alpha\Downarrow}(\tau_1^{\beta}, \tau_*^{\beta}) = \chi^{*+},$$

that is $h(\mathcal{S}t) = \chi^{*+}$ and heights of these steps amounts strictly up to χ^{*+} ;

(iv) for every maximal unit step $[\gamma_{\tau_1}^{<\alpha\Downarrow}, \gamma_{\tau_2}^{<\alpha\Downarrow}[$ below $\alpha\Downarrow$ the corresponding triple (τ_1, τ_*, τ_2) is the value of this function. ¹⁶⁾

Respectively, this stairway $\mathcal{S}t$ terminates in $v(\mathcal{S}t) = \alpha\Downarrow$, if its steps are disposed cofinally to $\alpha\Downarrow$, that is if there holds the property $H(\alpha\Downarrow)$:

$$\begin{aligned} \forall \gamma < \alpha\Downarrow \exists \beta < \chi^{*+} \exists \tau_1^{\beta}, \tau_*^{\beta}, \tau_2^{\beta}, (\gamma < \gamma_{\tau_1^{\beta}}^{<\alpha\Downarrow} < \gamma_{\tau_2^{\beta}}^{<\alpha\Downarrow} < \alpha\Downarrow \wedge \\ \wedge \mathcal{S}t(\beta) = (\tau_1^{\beta}, \tau_*^{\beta}, \tau_2^{\beta})). \end{aligned}$$

Lemma 11.3

For every matrix S of zero characteristic on a carrier $\alpha > \chi^*$:

(I) S on α is provided by some stairway $\mathcal{S}t$.

(II) This stairway $\mathcal{S}t$ terminates in $\alpha\Downarrow$, that is

$$v(\mathcal{S}t) = \alpha\Downarrow = \sup \{ \gamma_{\tau_2}^{<\alpha\Downarrow} : \exists \beta, \tau_1, \tau_* \ \mathcal{S}t(\beta) = (\tau_1, \tau_*, \tau_2) \}.$$

Proof I. Let us consider any carrier $\alpha_0 > \chi^*$ of the matrix S^0 of zero characteristic on α_0 and $\alpha^0 = \alpha_0\Downarrow$.

By lemma 8.5 5) there exist $\tau'_1, \tau'_2, \tau'_3$ such that below α^0

$$A_2^{<\alpha^0}(\tau'_1, \tau'_2, \tau'_3, \alpha S_f^{<\alpha^0}) \wedge \forall \tau'' \in]\tau'_1, \tau'_2] \ a_{\tau''}^{<\alpha^0} = 1 \wedge \alpha S_{\tau'_2}^{<\alpha^0} = S^0$$

and from here it follows $a_{\tau_2'}^{<\alpha^0} = 1$. Now let us consider S^0 on the carrier $\alpha_2 = \alpha_{\tau_2'}^{<\alpha^0}$ and $\alpha^2 = \alpha_2^\Downarrow$. By lemma 10.3 there are τ_*, τ_2'' such, that $\tau_1' < \tau_*' \leq \tau_2''$ and

$$A_{1.1}^{st<\alpha^2}(\tau_1', \tau_*', \tau_2'', \alpha S_f^{<\alpha^2}, a_f^{<\alpha^2}) \wedge A_{1.1}^{M<\alpha^2}(\tau_1', \tau_2'', \alpha S_f^{<\alpha^2}); \quad (11.10)$$

$$Od\alpha S_f^{<\alpha^2}(\tau_1', \tau_*') > Od(S^0). \quad (11.11)$$

Let us enumerate all the triples of ordinals $(\tau_1', \tau_*', \tau_2'')$ possessing property (11.10) without omission in the order of increasing of their first components, that is let us define the function

$$St = ((\tau_1^\beta, \tau_*^\beta, \tau_2^\beta))_\beta$$

with property (i) of the stairway presentation above for $\alpha^\Downarrow = \alpha_2$; statement (ii) comes after that from corollary 9.6 for $\alpha_1 = \alpha^2$. From here and (11.11) it follows that the ordinal $Od\alpha S_f^{<\alpha^2}(\tau_1^\beta, \tau_*^\beta)$ is strictly increasing along with β up to χ^{*+} ; in the opposite case it is possible to define below α^2 the upper bound of the set of these ordinals

$$\rho \in [Od\alpha S_f^{<\alpha^2}(\tau_1', \tau_*'); \chi^{*+}[$$

and then by lemma 4.6 [27] about spectrum type

$$\rho < Od(S^0)$$

in spite of (11.11). Thus $dom(St) = \chi^{*+}$; statement (iv) is obvious due to the construction of St .

So, statements (i)–(iv) are proved for the carrier α_2 of the matrix S^0 and on α_2 it is provided with this stairway. Then by lemma 5.11 [27] about informativeness the matrix S^0 is provided by some

stairway $\mathcal{S}t^0$ on its carrier α_0 as well, because this property is the *inner* property of S^0 (see comments before lemma 9.5).

II. Turning to statement (II) let us suppose that it is wrong and this $\mathcal{S}t^0$ terminates in some cardinal $v^0 < \alpha^0 = \alpha^{0\Downarrow}$:

$$v^0 = \sup \{ \gamma_{\tau_2}^{<\alpha^0} : \exists \beta, \tau_1, \tau_* \quad \mathcal{S}t^0(\beta) = (\tau_1, \tau_*, \tau_2) \};$$

evidently, v^0 belongs to $SIN_n^{<\alpha^0}$ and has the cofinality χ^{*+} .

The rest part of the reasoning relies on the method which may be called *sewing method*; here is its outline (below α^0):

Considering some cardinal v one can face with the situation when there are cardinals $\gamma_\tau < v$ disposed nearby this v and such that the function αS_f is assumed to be not defined for the corresponding τ ; therefore such cardinals $\gamma_\tau < v$ may be called “holes” in the set

$$v \cap \{ \gamma_\tau : \tau \in \text{dom}(\alpha S_f) \}.$$

In order to get over this situation and, nevertheless, to see αS_f be defined for such holes, one should perform the following two steps:

It should be discovered some α -matrix S on a carrier $\alpha \geq v$ of some characteristic a along with its generating disseminator $\check{\delta}^\rho < v$ and its base ρ so, that the interval $] \check{\delta}^\rho, v[$ contains such holes.

At the same time it should be discovered some cardinal

$$\gamma^\delta \in [\check{\delta}^\rho, v[\cap SIN_n$$

which along with S, ρ destroys the premise of the closing condition

$$\mathbf{K}^0(a, \gamma^\delta, \alpha, \rho)$$

or fulfils its conclusion and therefore holds it on, and cause of that by lemma 6.8 1) [27] γ^δ becomes also the admissible disseminator

of S on α with the same base. Moreover, one should see that this new disseminator is *admissible and nonsuppressed for every*

$$\gamma_\tau \in \{\gamma_\tau : \gamma^\delta < \gamma_\tau < v\},$$

because this condition trivially fulfils for γ_τ and, hence, the whole statement of admissibility

$$\alpha\mathbf{K}(a, \gamma^\delta, \gamma_\tau, \alpha, \rho, S)$$

holds for many carriers $\alpha > \gamma_\tau$ of the matrix S too. Therefore due to argument (1b.) the function αS_f is found to be defined on the whole set

$$\{\tau : \gamma^\delta < \gamma_\tau < v\},$$

and thus it happens the “sewing” of the interval $[\gamma^\delta, v[$ – it means, that this set includes in $\text{dom}(\alpha S_f)$ and this interval does not contain any holes in spite of the assumption.

The contradiction of this kind will help to advance on the proof of lemma 11.3, and, hence, the proof of theorem 2 at each critical stage.

So, let us consider as such v the cardinal $v^1 \in \text{SIN}_n^{<\alpha^0}$ which is χ^{*+} by order in $\text{SIN}_n^{<\alpha^0}$, that is the set

$$v^1 \cap \text{SIN}_n^{<\alpha^0}$$

has the order type χ^{*+} ; this cardinal $v^1 \leq v^0$ really exist due to $v^0 < \alpha^0$, $cf(v^0) = \chi^{*+}$.

Since $v^1 \in \text{SIN}_n^{<\alpha^0}$ and $cf(v^1) = \chi^{*+}$, there exist the δ -matrix S^1 of the characteristic a^1 reduced to χ^* and produced by the cardinal v^1 on the carrier $\alpha^1 < \alpha^0$ with the prejumpt cardinal $\alpha^{1\downarrow} = v^1$ and the generating eigendisseminator $\delta^1 = \check{\delta}^{S^1} < v^1$ with the base $\rho^1 = \rho^{S^1}$ by lemma 6.13 [27] (used for $m = n + 1$, $\alpha_0 = v^1$, $\alpha_1 = \alpha^0$) and the function

$$f(\beta) = OT(\beta \cap \text{SIN}_n^{<v^1});$$

we shall consider the minimal α^1 for some definiteness.

We shall see, that it provides the contradiction: there arises certain set

$$T^{v^1} = \{\tau : \gamma < \gamma_\tau^{v^1} < v^1\} \subseteq \text{dom}(\alpha S_f^{<v^1})$$

meeting all the conditions of theorem 2 (for v^1 instead of α_1), contrary to the minimality of α_1^* ; this contradiction shows, that in fact the stairway $\mathcal{S}t^0$ terminates in the cardinal $v^0 = \alpha^0 = \alpha_0^\Downarrow$. This effect will be achieved by the sewing method applied to v^1 . First, there arises the covering of the interval $[\delta^1, v^1[$ by maximal blocks (below v^1). Suppose it is wrong, then there exist some cardinal

$$\gamma^1 \in \text{SIN}_n^{<v^1} \cap]\delta^1, v^1[$$

which does not belong to any block (below v^1). Therefore this γ^1 can serve as the disseminator

$$\tilde{\delta}^1 = \gamma^1$$

with the same base ρ^1 by lemma 6.8 [27] (for $m = n + 1$) admissible for every $\gamma_\tau \in]\tilde{\delta}^1, v^1[$, since there holds the closing Δ_1 -condition $\mathbf{K}^0(\alpha^1, \delta^1, \alpha^1, \rho^1)$

$$\begin{aligned} (a^1 = 0 \rightarrow \forall \tau'_1, \tau''_1, \tau'_2, \tau'_3, \eta' < \alpha^{1\Downarrow} [\gamma_{\tau'_1}^{<\alpha^{1\Downarrow}} \leq \tilde{\delta}^1 < \gamma_{\tau'_3}^{<\alpha^{1\Downarrow}} \wedge \\ \wedge A_4^{M\triangleleft\alpha^{1\Downarrow}}(\tau'_1, \tau''_1, \tau'_2, \tau'_3, \eta', \alpha S_f^{<\alpha^{1\Downarrow}}, a_f^{<\alpha^{1\Downarrow}}) \rightarrow \eta' < \rho^1 \vee \rho^1 = \chi^{*+}]) \end{aligned} \quad (11.12)$$

due to the failure of its premise $A_4^{M\triangleleft\alpha^{1\Downarrow}}$. Now the sewing method works: for every $\gamma_\tau \in]\tilde{\delta}^1, v^1[$ and for

$$\gamma_{\tau^n} = \sup\{\gamma \leq \gamma_\tau : \gamma \in \text{SIN}_n^{<v^1}\}$$

there holds the Π_{n-2} -proposition $\varphi(a^1, \tilde{\delta}^1, \gamma_{\tau^n}, \gamma_\tau, \alpha^1, \rho^1, S^1)$:

$$\gamma_\tau < \alpha^1 \wedge \text{SIN}_n^{<\alpha^{1\Downarrow}}(\gamma_{\tau^n}) \wedge \alpha \mathbf{K}_{n+1}^\exists(a^1, \tilde{\delta}^1, \gamma_\tau, \alpha^1, \rho^1, S^1)$$

stating, that S^1 on α^1 is admissible for γ_τ along with the same $a^1, \tilde{\delta}^1, \rho^1$. Then the SIN_{n-1} -cardinal $\gamma_{\tau+1}$ restricts the Σ_{n-1} -proposition

$$\exists \alpha(\gamma_\tau < \alpha \wedge \varphi(a^1, \tilde{\delta}^1, \gamma_\tau, \gamma_\tau, \alpha, \rho^1, S^1)) \quad (11.13)$$

and therefore in $] \gamma_\tau, \gamma_{\tau+1}[$ there appear many admissible for γ_τ carriers α with this property (11.13) and it provides the contradiction below v^1 :

if $a^1 = 1$, then they are all nonsuppressed by definition; if $a^1 = 0$, then they are all nonsuppressed for γ_τ by argument (1b.) above; anyhow the sewing of v^1 happens: there appears some set T^{v^1} of the function $\alpha S_f^{<v^1}$ definiteness with properties (i)-(iii) from theorem 2 (for $\alpha_1 = v^1$), contrary to the minimality of α_1^* .

So, the interval $[\tilde{\delta}^1, v^1[$ is covered by maximal blocks below v^1 and there holds

$$A_{5.1}^{sc \triangleleft v^1}(\gamma^m, \alpha S_f^{<v^1}, a_f^{<v^1})$$

stating the covering of $[\gamma^m, v^1[$ by the maximal blocks below v^1 and the minimality of the cardinal γ^m with this property (remind definition 8.1 2.1a., 2.1b.).

The rest part of this lemma 11.3 proof is conducted below v^1 and the upper indices $< v^1, \triangleleft v^1$ and notations of the functions $\alpha S_f^{<v^1}, a_f^{<v^1}$ will be dropped as usual (when the context will point out them clearly).

Here the final contradiction of this proof comes:

this covering cannot be η -bounded, and at the same time it have to be η -bounded below v^1 (remind definition 11.2 2) for $\alpha^\Downarrow = v^1$).

Really, this covering cannot be η -bounded, because in the opposite case there exist some constant type of its maximal blocks, disposed cofinally to v^1 . The minimal type η^1 of such types is obviously defined below $v^1 = \alpha^{1\Downarrow}$ and by lemma 4.6 [27] about spectrum type there holds

$$\eta^1 < Od(S^1) < \rho^1.$$

Here again the sewing method works. Let $[\gamma_{\tau_1}, \gamma_{\tau_2}[$ be the maximal block in $[\tilde{\delta}^1, v^1[$ of this type η^1 with the minimal left end γ_{τ_1} , then SIN_n -cardinal γ_{τ_1} again can serve as the disseminator $\tilde{\delta}^{1'} = \gamma_{\tau_1}$ for S^1 on α^1 with the same base ρ^1 .

And again proposition (11.12) holds (where $\tilde{\delta}^1$ should be replaced with $\tilde{\delta}^{1'}$), but now because S^1 on α^1 leans on $\tilde{\delta}^{1'}$ very strongly: there is the single maximal block $[\gamma_{\tau'_1}^{<\alpha^{1\downarrow}}, \gamma_{\tau'_3}^{<\alpha^{1\downarrow}}[$, that is precisely $[\gamma_{\tau_1}, \gamma_{\tau_2}[$ of the type η^1 , which contains $\tilde{\delta}^{1'}$ and which meet its conclusion $\eta^1 < \rho^1$. And again there holds (11.13) for every $\gamma_\tau \in]\tilde{\delta}^{1'}, v^1[$ and there arises some set T^{v^1} in the same way, contrary to the minimality of α_1^* .

Thus the covering of $[\gamma^m, v^1[$ must not be η -bounded, that is the types of its maximal blocks have to nondecrease up to χ^{*+} substantially (below v^1):

$$\forall \eta < \chi^{*+} \exists \gamma' < v^1 \forall \tau'_1, \tau'_2, \eta' (\gamma' < \gamma_{\tau'_2} \leq v^1 \wedge \quad (11.14)$$

$$\wedge A_4^{Mb}(\tau'_1, \tau'_2, \eta', \alpha S_f^{<v^1}, a_f^{<v^1}) \rightarrow \eta < \eta')$$

But it provides the contradiction: infinitely many values of the matrix function $\alpha S_f^{<v^1}$ become suppressed (below v^1), though they are nonsuppressed by definition 8.3 of this matrix function.

To see it one should apply the reasoning mode, that may be called the “blocks cut-off from the right” and that consists in the “shortening” of the blocks, which are “too long”, from their right ends.

This method works here quite well because this covering is not η -bounded and therefore it is working on these blocks as the suppressing covering – more precisely, holding the condition (see definition 8.1 2.4 for $X_1 = \alpha S_f^{<v^1} | \tau$, $X_2 = a_f^{<v^1} | \tau$):

$$A_{5.4}^{sc}(\gamma_\tau, \eta^*, \alpha S_f^{<v^1} | \tau, a_f^{<v^1} | \tau)$$

for some $\gamma^m, \gamma^*, \gamma^1, \eta^*$ and for infinitely many cardinals γ_τ disposed cofinally to v^1 .

The cardinal γ^m here is already defined above as the minimal of blocks ends covering of the cardinal v^1 .

Next, the ordinals γ^*, η^* can be defined here in different ways, for instance, as limit points of the following sequences due to (11.14) below v^1 (the notations $\alpha S_f^{<v^1}, a_f^{<v^1}$ will be dropped):

$$\begin{aligned}
\gamma_0 &= \gamma^m; \\
\eta_i &= \sup \{ \eta : \exists \tau_1, \tau_2 (\gamma_{\tau_1} < \gamma_{\tau_2} < \gamma_i \wedge A_4^{Mb}(\tau_1, \tau_2, \eta)) \}; \\
\gamma_{i+1} &= \min \{ \gamma : \exists \tau_1, \tau_2, \eta (\gamma_i < \gamma_{\tau_1} < \gamma_{\tau_2} = \gamma \wedge \eta_i < \eta \wedge \\
&\quad \wedge A_4^{Mb}(\tau_1, \tau_2, \eta) \wedge \forall \tau'_1, \tau'_2, \eta' (\gamma \leq \gamma_{\tau'_1} < \gamma_{\tau'_2} \wedge \\
&\quad \wedge A_4^{Mb}(\tau'_1, \tau'_2, \eta') \rightarrow \eta \leq \eta') \}; \\
\eta^* &= \sup_{i \in \omega_0} \eta_i; \quad \gamma^* = \sup_{i \in \omega_0} \gamma_i.
\end{aligned} \tag{11.15}$$

Since this covering is not η -bounded, there exist the maximal block in $[\gamma^*, v^1[$

$$[\gamma_{\tau_1^*}, \gamma_{\tau_2^*}[\text{ of the greater type } \eta^{*1} > \eta^*;$$

and one should take such block just with the *minimal* left end $\gamma_{\tau_1^*} > \gamma^*$. Therefore this block includes the initial subinterval

$$[\gamma_{\tau_1^*}, \gamma^{*1}[\text{ exactly of the type } \eta^*,$$

which is also the block (not maximal) with right end $\gamma^{*1} \in SIN_n$, $\gamma^{*1} = \gamma_{\tau_2^{*1}}$.

Hence, there exist the matrix

$$S^{*1} = \alpha S_{\tau_2^{*1}}$$

on its carrier $\alpha^{*1} = \alpha_{\tau_2^{*1}}$ of the characteristic $a^{*1} = a_{\tau_2^{*1}}$, which is admissible and *nonsuppressed* for γ^{*1} along with its disseminator $\tilde{\delta}^{*1} = \tilde{\delta}_{\tau_2^{*1}}$ with the base $\rho^{*1} = \rho_{\tau_2^{*1}}$ by definition

(all it below v^1).

But at the same time this matrix S^{*1} and all its accessories are, vice versa, suppressed for γ^{*1} because there holds the suppression condition $A_{5,0}^{S,0}$ for them below v^1 (remind definition 8.1 2.6 and also (11.5)), which has now the form:

$$\begin{aligned}
a^{*1} &= 0 \wedge SIN_n^{<v^1}(\gamma^{*1}) \wedge \rho^{*1} < \chi^{*+} \wedge \sigma(\chi^*, \alpha^{*1}, S^{*1}) \wedge \\
&\wedge \exists \eta^*, \tau < \gamma^{*1} \left(\gamma^{*1} = \gamma_\tau^{<v^1} \wedge A_{5,4}^{sc \triangleleft v^1}(\gamma^{*1}, \eta^*, \alpha S_f^{<v^1} | \tau, a_f^{<v^1} | \tau) \wedge \right. \\
&\wedge \forall \tau' \left(\tau < \tau' \wedge SIN_n^{<v^1}(\gamma_{\tau'}^{<v^1}) \rightarrow \right. \\
&\rightarrow \exists \alpha', S' \left[\gamma_{\tau'}^{<v^1} < \alpha' < \gamma_{\tau'+1}^{<v^1} \wedge SIN_n^{<\alpha'^\Downarrow}(\gamma_{\tau'}^{<v^1}) \wedge \sigma(\chi^*, \alpha', S') \wedge \right. \\
&\quad \left. \left. \wedge A_{5,5}^{sc}(\gamma^{*1}, \eta^*, \alpha'^\Downarrow, \alpha S_f^{<\alpha'^\Downarrow}, a_f^{<\alpha'^\Downarrow}) \right] \right) \Big).
\end{aligned} \tag{11.16}$$

Here actually $a^{*1} = 0$ by lemma 11.1; $SIN_n^{<v^1}(\gamma^{*1})$ by definition; $\rho^{*1} < \chi^{*+}$ since v^1 is χ^{*+} by order in $SIN_n^{<v^1}$; $\sigma(\chi^*, \alpha^{*1}, S^{*1})$ due to the admissibility of S^{*1} on α^{*1} for γ^{*1} ; $A_{5,4}^{sc \triangleleft v^1}$ holds since types of γ^* covering are nondecreasing up to η^* substantially by (11.15); and the maximal blocks from the interval $[\gamma^*, \gamma_{\tau_1}^*]$ have the constant type η^* due to the minimality of $\gamma_{\tau_1}^*$ and it remains only to check the condition $A_{5,5}^{sc}$ from (11.16). To this end one should apply the usual restriction argument:

Every block $[\gamma_{\tau_1}, \gamma_{\tau_2}[$ in $[\gamma^*, v^1[$ has the type $\eta \geq \eta^*$ due to (11.15) and that is why there holds the following Π_{n-2} -formula $\psi(\gamma^{*1}, \eta^*, \alpha^1, S^1, \alpha S_f^{<\alpha^1 \Downarrow}, a_f^{<\alpha^1 \Downarrow})$:

$$\sigma(\chi^*, \alpha^1, S^1) \wedge A_{5,5}^{sc}(\gamma^{*1}, \eta^*, \alpha^1 \Downarrow, \alpha S_f^{<\alpha^1 \Downarrow}, a_f^{<\alpha^1 \Downarrow}),$$

where $A_{5.5}^{sc}$ is the Δ_1 -formula (see definition 8.1 2.5):

$$\begin{aligned} \forall \gamma' \Big(\gamma^{*1} \leq \gamma' < \alpha^{1\Downarrow} \rightarrow \exists \tau'_1, \tau'_2, \eta' \Big(\gamma_{\tau'_1}^{<\alpha^{1\Downarrow}} \leq \gamma' < \gamma_{\tau'_2}^{<\alpha^{1\Downarrow}} \wedge \\ \wedge A_4^{M<\alpha^{1\Downarrow}}(\tau'_1, \tau'_2, \eta', \alpha S_f^{<\alpha^{1\Downarrow}}, a_f^{<\alpha^{1\Downarrow}}) \wedge \eta' \geq \eta^* \Big) \Big). \end{aligned}$$

Now one should take any $\gamma_{\tau'} > \gamma^{*1}$, $\gamma_{\tau'} \in SIN_n^{<v^1}$; by lemma 3.2 [27] about restriction SIN_{n-1} -cardinal $\gamma_{\tau'+1}$ restricts the Σ_{n-1} -proposition $\exists \alpha' \psi_1(\gamma^{*1}, \eta^*, \alpha', \gamma_{\tau'})$:

$$\begin{aligned} \exists \alpha' \exists S' \Big[\gamma_{\tau'} < \alpha' \wedge SIN_n^{<\alpha'^\Downarrow}(\gamma_{\tau'}) \wedge \\ \wedge \psi(\gamma^{*1}, \eta^*, \alpha', S', \alpha S_f^{<\alpha'^\Downarrow}, a_f^{<\alpha'^\Downarrow}) \Big], \end{aligned}$$

and some carrier α' of matrix S' with this property appears in $]\gamma_{\tau'}, \gamma_{\tau'+1}[$. Thus below v^1 there holds the proposition:

$$\forall \tau' (\tau_2^{*1} < \tau' \wedge SIN_n^{<v^1}(\gamma_{\tau'}) \rightarrow \exists \alpha' < \gamma_{\tau'+1} \psi_1^{<v^1}(\gamma^{*1}, \eta^*, \alpha', \gamma_{\tau'}));$$

as the result the whole suppression condition (11.16) is carried out for the matrix S^{*1} on its carrier α^{*1} and it cannot be the value of the matrix function $\alpha S_f^{<v^1}$ below v^1 contrary to the assumption.

—

Here the final part of theorem 2 proof comes to its close. Again all reasonings will be relativized to α_1^* and therefore the upper indices $< \alpha_1^*$, $\triangleleft \alpha_1^*$ and the denotations of the functions $\alpha S_f^{<\alpha_1^*}$, $a_f^{<\alpha_1^*}$ will be dropped.

By the supposition this theorem fails for the minimal cardinal α_1^* , so there exist τ_2^*, τ_3^* such that there holds

$$A_2(\tau_1^*, \tau_2^*, \tau_3^*)$$

where, remind, τ_1^* is the minimal ordinal in $T^{\alpha_1^*}$ and where τ_2^* is the minimal ordinal at which the monotonicity of αS_f on $T^{\alpha_1^*}$ is broken; let us consider any arbitrary great ordinal τ_3^* from the set

$$Z^* = \{\tau : \gamma_2^* < \gamma_\tau < \alpha_1^* \wedge \gamma_\tau \in SIN_n\},$$

and consider the corresponding cardinals

$$\gamma_i^* = \gamma_{\tau_i^*}, \quad i = \overline{1, 3},$$

and the matrix $S^{*2} = \alpha S_{\tau_2^*}$ on the carrier $\alpha_{\tau_2^*}$ of the characteristic $a^{*2} = a_{\tau_2^*}$ with the prejump cardinal $\alpha^{*2} = \alpha_{\tau_2^*}^\Downarrow$ and its generating eigendisseminator $\check{\delta}^{*2} = \check{\delta}_{\tau_2^*}^S$. But the main role will be played by the matrix

$$S^{*3} = \alpha S_{\tau_3^*} \text{ on the carrier } \alpha_{\tau_3^*}$$

for this $\tau_3^* \in Z^*$ with the prejump cardinal $\alpha^{*3} = \alpha_{\tau_3^*}^\Downarrow$, and the generating and floating disseminators

$$\check{\delta}^{*3} = \check{\delta}_{\tau_3^*}, \quad \tilde{\delta}^{*3} = \tilde{\delta}_{\tau_3^*} \text{ with the base } \rho^{*3} = \rho_{\tau_3^*}.$$

From lemma 11.1 (for $\tau_1^*, \tau, \alpha S_\tau, \alpha_\tau, \alpha_1^*$ as $\tau_1, \tau_2, S^2, \alpha^2, \alpha_1$) it follows:

$$\forall \tau \in Z^* \quad (a_\tau = 0 \wedge \tilde{\delta}_\tau = \gamma_1^*). \quad (11.17)$$

Now the following cases come:

Case 1. $a^{*2} = 1$. Then by lemma 10.3 there exist $\tau_1^{*'}$ such that

$$A_3(\tau_1^*, \tau_1^{*'}, \tau_2^*, \tau_3^*). \quad (11.18)$$

where, remind, the matrix $\alpha S_{\tau_1^{*'}}^*$ has unit characteristic on its carrier $\alpha_{\tau_1^{*'}}^*$.

From this moment one should use only the ordinal τ_3^* such that the interval

$$[\gamma_1^*, \gamma_{\tau_3^*}^*[$$

has the type

$$\eta^{*3} > Od(\alpha S_{\tau_1^{*'}}^*);$$

the existence of such ordinal τ_3^* follows from conditions (i), (iii) of this theorem 2.

Let us consider the following subcases:

Case 1a. Suppose that

$$\alpha S_{\tau_1^{*'}}^* \triangleleft \rho^{*3}.$$

But it is excluded by the restriction-and-extension, providing the following argument that may be called the “stairway cut-off from above” and that consists in cutting of heights of stairway steps.

The matrix $\alpha S_{\tau_1^{*'}}^*$ of unit characteristic has the admissible carrier

$$\alpha_{\tau_1^{*'}}^* \in]\tilde{\delta}_3^*, \gamma_3^*[,$$

since $\tilde{\delta}_3^* = \gamma_1^*$. By lemma 3.2 [27] this matrix receives its carriers of unit characteristic disposed cofinally to $\check{\delta}^{*3} \leq \tilde{\delta}^{*3}$ and therefore there holds the Π_{n+1} -proposition below $\check{\delta}^{*3}$:

$$\forall \gamma \exists \gamma^1 > \gamma \exists \delta, \alpha, \rho (SIN_{n-1}(\gamma^1) \wedge \alpha \mathbf{K}(1, \delta, \gamma^1, \alpha, \rho, \alpha S_{\tau_1^{*'}}^*)),$$

and due to lemma 6.6 [27] (for $m = n + 1$) this disseminator extends this proposition up to the prejump cardinal α^{*3} and, so, $\alpha S_{\tau_1^{*'}}^*$ receives its admissible carriers of unit characteristic below α^{*3} disposed cofinally up to α^{*3} .

By lemma 11.3 there exist the stairway St terminating in α^{*3} , but by lemma 9.5 2b. (about stairway cut-off from above, where

α_1, S^0 are replaced with $\alpha^{*3}, \alpha S_{\tau_1^{*'}})$ this stairway $\mathcal{S}t$ is impossible, because all steps $\mathcal{S}t(\beta)$ of this stairway receives heights less then the ordinal

$$Od(\alpha S_{\tau_1^{*'}}) < \chi^{*+},$$

though they amount up to χ^{*+} by definition.

Case 1b. So,

$$\rho^{*3} \leq Od(\alpha S_{\tau_1^{*'}}).$$

But recall that here is used the type

$$\eta^{*3} > Od(\alpha S_{\tau_1^{*'}})$$

of the block $[\gamma_1^*, \gamma_3^*]$.

Due to (11.18) there exist ordinals τ_3', η_3' , such that for $\alpha^{*3} = \alpha_{\tau_3^*}^{\downarrow}$ there holds

$$A_4^{Mb \triangleleft \alpha^{*3}}(\tau_1^*, \tau_1^{*'}, \tau_2^*, \tau_3', \eta_3'). \quad (11.19)$$

These ordinals $\tau_1^*, \tau_1^{*'}, \tau_2^*, \tau_3', \eta_3'$ are uniquely defined through χ^* , $\gamma_1^* = \tilde{\delta}^{*3}$ below α^{*3} and it is not hard to see, that

$$\tau_3^* \leq \tau_3', \quad \eta^{*3} \leq \eta_3'.$$

From the admissibility of S^{*3} on $\alpha_{\tau_3^*}$ and lemma 8.5 6) it comes now the proposition $\mathbf{K}^0(a^{*3}, \tilde{\delta}^{*3}, \alpha_{\tau_3^*}, \rho^{*3})$:

$$a^{*3} = 0 \longrightarrow \forall \tau_1', \tau_1'', \tau_2', \tau_3', \eta' \left[\gamma_{\tau_1'}^{< \alpha^{*3}} \leq \tilde{\delta}^{*3} < \gamma_{\tau_3'}^{< \alpha^{*3}} \wedge \right.$$

$$\left. \wedge A_4^{Mb \triangleleft \alpha^{*3}}(\tau_1', \tau_1'', \tau_2', \tau_3', \eta', \alpha S_f^{< \alpha^{*3}}, a_f^{< \alpha^{*3}}) \rightarrow \eta' < \rho^{*3} \vee \rho^{*3} = \chi^{*+} \right].$$

Here again the block $[\gamma_{\tau_1'}^{< \alpha^{*3}}, \gamma_{\tau_3'}^{< \alpha^{*3}}]$ is defined through $\tilde{\delta}^{*3} = \gamma_1^*$ uniquely and therefore

$$\tilde{\delta}^{*3} = \gamma_1^* = \gamma_{\tau_1'}^{< \alpha^{*3}}, \quad \eta^{*3} \leq \eta_3' = \eta'.$$

From here, from (11.17) and (11.19) it follows that S^{*3} on $\alpha_{\tau_3^*}$ leans on $\widetilde{\delta}^{*3}$ very strongly and

$$\alpha S_{\tau_1^{*'}} \triangleleft \rho^{*3}$$

contrary to the condition of this subcase.

Case 2. $a^{*2} = 0$. In this case below the prejump cardinal $\alpha^{*2} = \alpha_{\tau_2^*}^{\downarrow}$ there holds

$$\forall \gamma < \gamma_2^* \exists \tau (\gamma < \gamma_\tau^{<\alpha^{*2}} \wedge a_\tau^{<\alpha^{*2}} = 1).$$

This proposition obviously follows from lemma 11.3 because there exist some stairway St of unit steps disposed cofinally to α^{*2} . From here and lemma 10.4 comes the existence of the ordinal $\tau_1^{*'}$ for which (11.18) holds again (we preserve here the notation from case 1. for some convenience). It remains to repeat literally the reasoning coming after (11.18). The proof of theorem 2 came to its close.

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Now let us sum up.

All the reasonings were conducted in the system

$$ZF + \exists k \text{ (} k \text{ is weakly inaccessible cardinal);}$$

there was considered the countable standard model

$$\mathfrak{M} = (L_{\chi^0}, \in, =)$$

of the theory

$$ZF + V = L + \exists k \text{ (} k \text{ is weakly inaccessible cardinal),}$$

where any weakly inaccessible cardinal becomes strongly inaccessible.

In this model the matrix functions were considered; such function $\alpha S_f^{<\alpha_1}$ is defined on any nonempty set T^{α_1} , which exist for any sufficiently great cardinal $\alpha_1 < k$, $\alpha_1 \in SIN_n$ due to lemma 8.9. It provides the final contradiction: let us take any SIN_n -cardinal $\alpha_1 > \alpha\delta^*$ limit for $SIN_n \cap \alpha_1$ of the cofinality $cf(\alpha_1) \geq \chi^{*+}$ providing such nonempty set T^{α_1} with properties (i)–(iii) from theorem 2, then the function $\alpha S_f^{<\alpha_1}$ is nonmonotone on this T^{α_1} by theorem 1 and at the same time is monotone on this set by theorem 2.

This contradiction ends the Main theorem proof.

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12 Some Consequences

Let us return to the beginning in the Introduction [27] where the diverse well-known interrelations between Large Cardinal Hypotheses, Axiom of Determinacy, the regular set properties, etc. were mentioned (see Drake [8], Kanamori [9]). Here we shall set forth some easy consequences of such results and the Main theorem.

I. Large Cardinal Hypotheses

The hierarchy of large cardinals arranges them “by the extent of inaccessibility” and bases on the (weakly) inaccessible cardinals. The existence of some of them (Mahlo cardinals, weakly compact, etc.) is directly forbidden by the Main theorem. Hence, there are no cardinals possessing stronger partition properties, for example, indescribable, Ramsey, Erdős cardinals and others; the measurable cardinals do not exist also since they are Ramsey cardinals. In some cases in the proof of the inconsistency of Large Cardinal Hypotheses AC can be used, but it is possible to avoid it, deriving from such hypotheses the existence of model of $ZFC + \exists$ inaccessible cardinal (see Silver [29] for example).

We omit the reformulation of this results in terms of filters, trees, infinitary languages etc.

II. Singular Cardinals. Sharps

By the nonexistence of inaccessibles every uncountable limit cardinal is singular. It is known that in ZFC every successive cardinal is regular. Hence any cardinal is singular *iff* it is uncountable and limit.

It is well-known the remarkable result of Jensen: the negation of the Inaccessible Cardinal Hypothesis implies the Singular Cardinal Hypothesis (see also Stern [30]). The non-existence of inner model with a measurable cardinal implies the Covering Lemma

for the core model K : for every uncountable $X \subseteq On$ there is $Y \in K$ such that $X \subseteq Y$ and $|X| = |Y|$.

From here the Singular Cardinal Hypothesis comes out (Dodd, Jensen [31, 32]). So, the nonexistence of inaccessibles provides this Covering Lemma and the Singular Cardinal Hypothesis.

This situation sheds a new light on the problem of sharps.

It is well-known that the existence of 0^\sharp implies the existence of cardinals inaccessible in L (Gitik, Magidor, Woodin [33]). Hence 0^\sharp does not exist; this result implies the Covering Lemma for L by the celebrated theorem of Jensen (see Devlin, Jensen [34]).

From here and the famous result of Kunen, establishing equivalence of the existence of elementary embedding $L \prec L$ and the existence of 0^\sharp , it follows that there are no elementary embeddings $L \prec L$ and, next, there are no elementary embeddings $L_\alpha \prec L_\beta$ with a critical point less than $|\alpha|$.

III. Axiom of Determinacy

It is known that AD implies certain Large Cardinal Hypotheses. For example, Solovay had proved, that AD implies the measurability of the cardinal ω_1 ; cardinals ω_2 , $\omega_{\omega+1}$, $\omega_{\omega+2}$ also are measurable (see also Kleinberg [35], Mignone [36]). Besides that, AD implies that cardinals ω_1 , ω_2 are δ -supercompact for inaccessible cardinal δ (Becker [37]). Mycielski [38] had proved that the consistency of

$$ZF + AD$$

implies the consistency of

$$ZFC + \exists \text{ inaccessible cardinal.}$$

Hence, AD is inconsistent and it is possible to define more exactly its inconsistent part:

Namely, $AD(\Sigma_2^1)$ is equiconsistent with the Measurable Cardinal Hypothesis (Louveau [39]). So, there exist undetermined Σ_2^1 -games.

Also, using $AC_\omega({}^\omega\omega)$, one can prove $\neg Det(\Pi_1^1)$ (see Kanamori [9]). Therefore in

$$ZF + AC_\omega({}^\omega\omega)$$

there exist undetermined Π_1^1 -games. This result can be hardly improved because every Δ_1^1 -game is determined (Martin [40]).

IV. Continual Sets

There are well-known series of outstanding results linking the regular set properties with large cardinals (by the relative consistency). For example, Shelah [41] established the necessity of inaccessible cardinals for the proposition that every set of reals is Lebesgue measurable; the measurability of Σ_3^1 -sets implies the inaccessibility of ω_1 in L (see also Raisonnier [42]). From here it follows the existence of unmeasurable Σ_3^1 -set of reals. Analogously, the consistence of

$$ZF + DC +$$

$$+ (\text{every uncountable set } \subseteq {}^\omega\omega \text{ has the perfect set property})$$

cannot be proved without the Measurable Cardinal Hypothesis (see Mycielsky [38]). Besides that, the following hypotheses are equiconsistent over ZF :

- 1) $AC + \exists$ inaccessible cardinal;
- 2) $DC +$ every uncountable set of reals has the perfect set property;
- 3) ω_1 is regular $+ \omega_1 \not\leq 2^{\omega_0}$;
- 4) ω_1 is regular $+ \forall a \in {}^\omega\omega \left(\omega_1^{L[a]} < \omega_1 \right)$

(see Solovay [26], Specker [43], Levy [21]). So, DC implies the

existence of a set of reals without the perfect set property and the regularity of ω_1 implies

$$\omega_1 \leq 2^{\omega_0} \wedge \omega_1^{L[a]} \not\leq \omega_1$$

for some $a \in {}^\omega\omega$. Also it is well-known that the Weakly Compact Cardinal Hypothesis is equiconsistent (over ZFC) to the statement about the regular properties of all continual sets in $ZF+MA$ (Harrington, Shelah [44]). Hence, the Main theorem implies the existence of uncountable sets $\subseteq \omega_\omega$ without regular properties. These results should be made more precise; for example, Solovay [28] proved that for any $a \in {}^\omega\omega$ $\omega_1^{L[a]} < \omega_1$ is equivalent to the perfect set property of every $\Pi_1^1(a)$ -set of reals. From here it comes now that the regularity of ω_1 implies the existence of $\Pi_1^1(a)$ -set of reals without the perfect set property for some $a \in {}^\omega\omega$.

V. Martin Axiom

The consequences of the results of Harrington, Shelah [44] and the Main theorem mentioned above should be formulated more exactly, for example, in the following way:

MA implies the existence of Δ_3^1 -sets without the Baire property and unmeasurable Σ_3^1 -sets $\subseteq {}^\omega\omega$.

Kanovei [45] had proved in

$$ZFC + MA + |R| > \omega_1 + \forall x \subseteq \omega_1 \quad \omega_1^{L[x]} = \omega_0$$

the consistency of

$$ZFC + \exists \text{ inaccessible cardinal.}$$

Hence, MA is inconsistent in

$$ZFC + |R| > \omega_1 + \forall x \subseteq \omega_1 \quad \omega_1^{L[x]} = \omega_0.$$

The host of other consequences of the Main theorem is too long to enumerate; so, the author intends to represent the more detailed analysis of such consequences in publications forthcoming.

Comments

⁹⁾ p. 31. This last refinement is not necessary and the Main theorem proof can be conducted without it, but still it should be accepted in order to shorten the reasoning forthcoming.

¹⁰⁾ p. 42. This definition has been presented earlier (Kiselev [15–17]) by means of the uniform text, but now here it is split into parts to clarify its structure.

¹¹⁾ p. 45. This closure condition \mathbf{K}^0 operates as the condition \mathbf{K}^0 used earlier (Kiselev [15–17]), but in the more managing way, because now it manages quite well without the subformula $(\gamma_{\tau'_3}^{<\alpha_\chi} = \gamma \rightarrow \lim(\gamma))$, which caused the significant proof complication.

¹²⁾ p. 47. This notion could be introduced in the nonrelativized form for $\alpha_1 = k$ as well, but it is not used in what follows; besides that, in this form it requires the more complicated non-elementary language over L_k .

¹³⁾ p. 95. It can be proved, that here $\tilde{\delta}_{\tau_2}^{<\alpha_1} = \gamma_{\tau_1}^{<\alpha_1}$.

¹⁴⁾ p. 96. Again actually here $\tilde{\delta}_{\tau_2}^{<\alpha_1} = \gamma_{\tau_1}^{<\alpha_1}$.

¹⁵⁾ p. 98. In fact here $a_{\tau}^{<\alpha_1} \equiv 1$ on $] \tau_1, \tau_2[$; and again $\tilde{\delta}_{\tau_2}^{<\alpha_1} = \gamma_{\tau_1}^{<\alpha_1}$.

¹⁶⁾ p. 120. This condition is not necessary in what follows, but it is still accepted to make such stairway be single for some convenience.

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